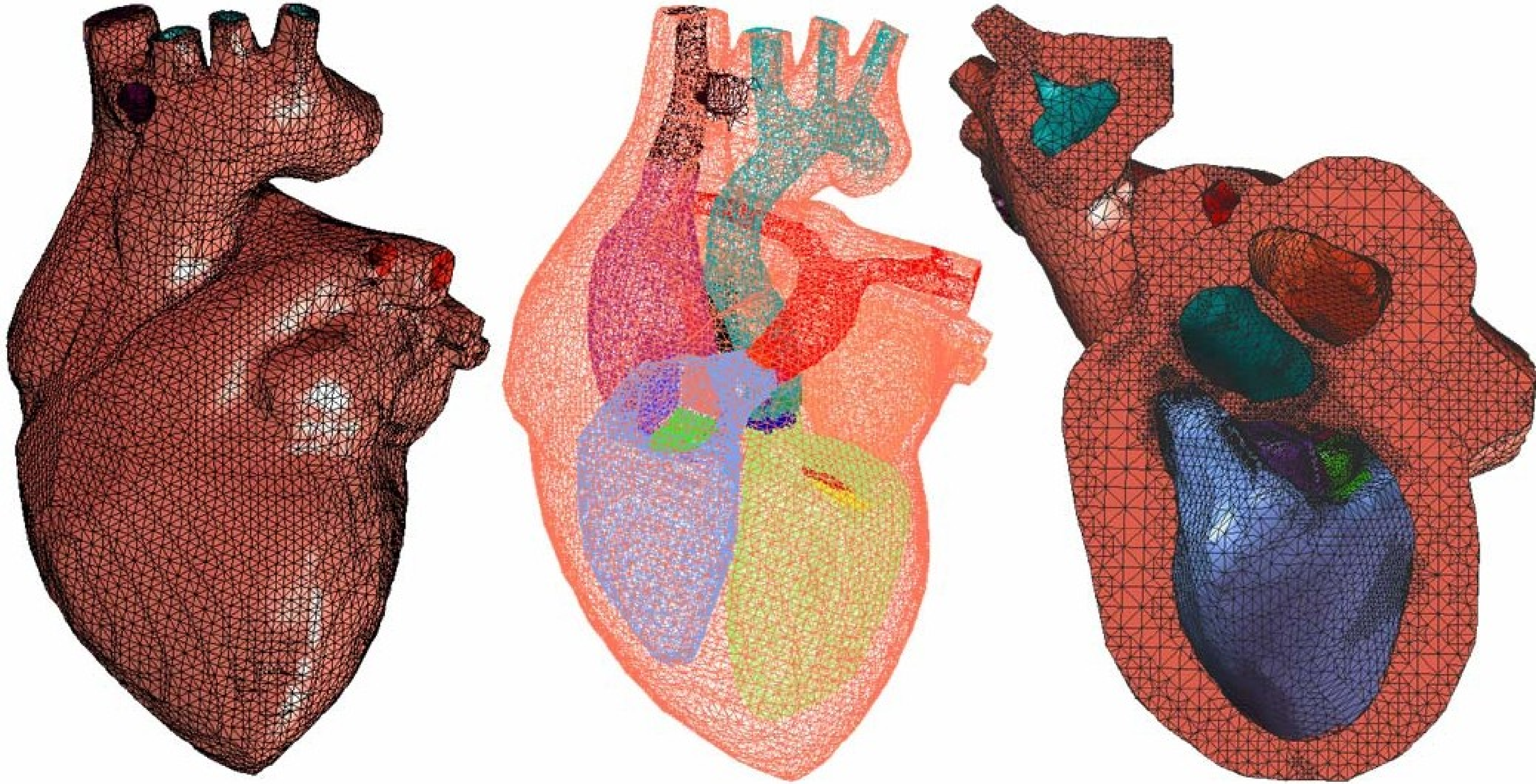
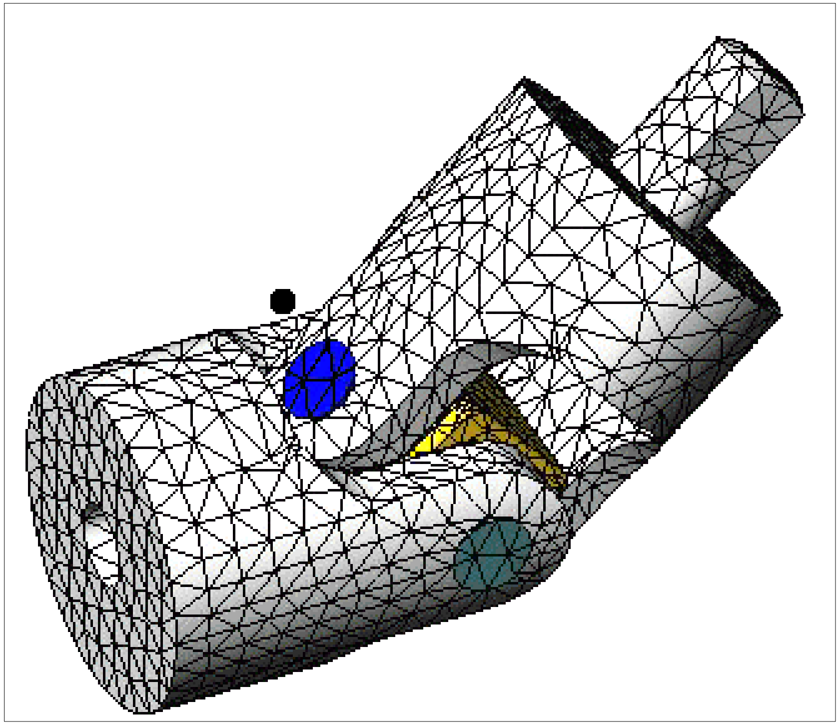
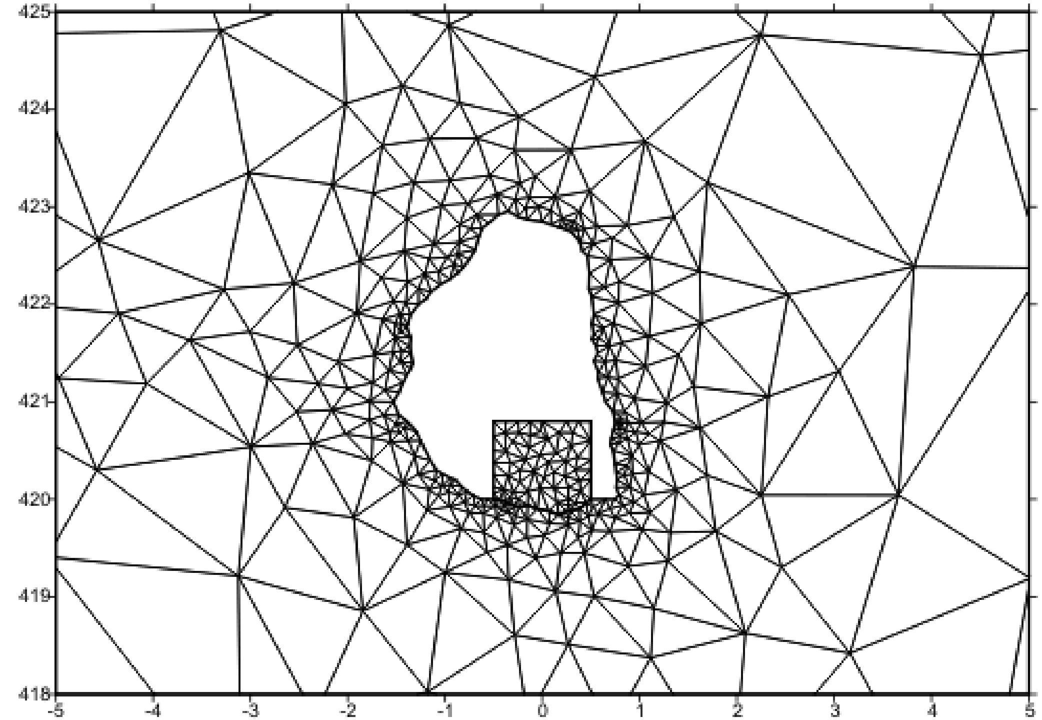
計算科学特論

有限要素法の連立方程式







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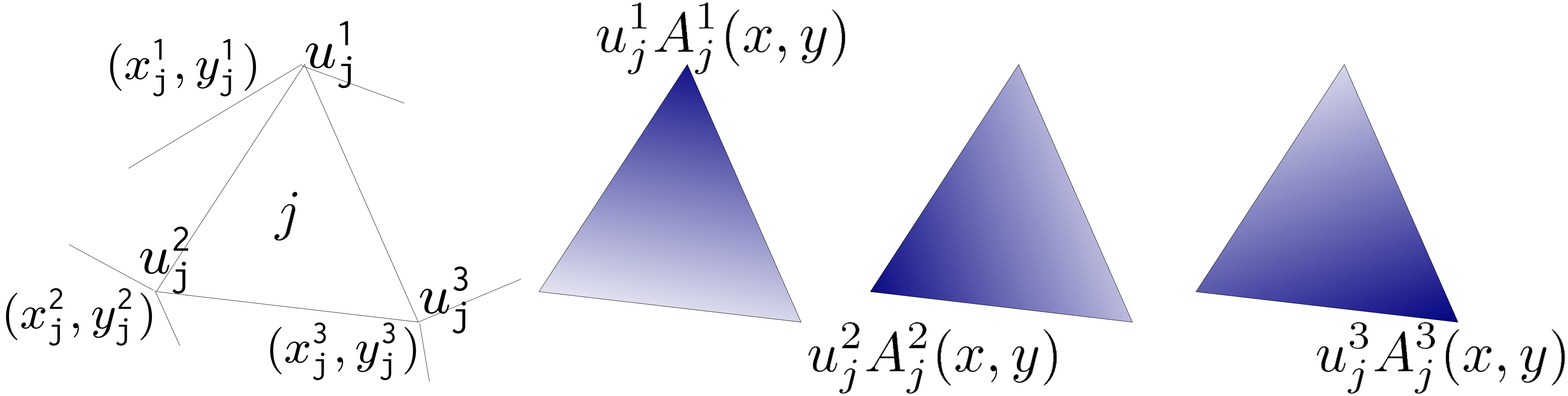
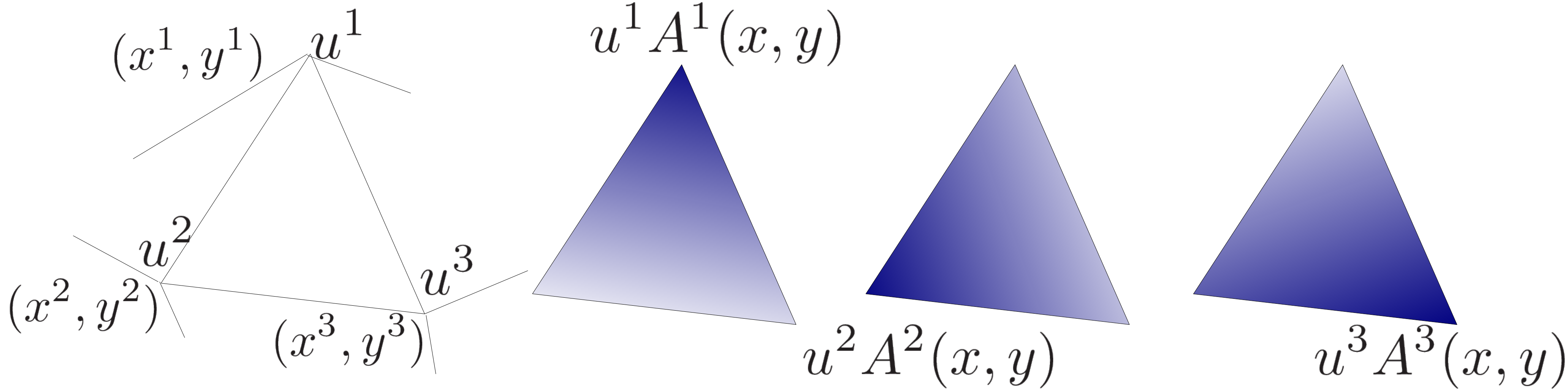


(x_j^1, y_j^1)
u_j^1
j
u_j^2
(x_j^2, y_j^2)
(x_j^3, y_j^3)
u_j^3
$u_j^1 A_j^1(x, y)$
$u_j^2 A_j^2(x, y)$
$u_j^3 A_j^3(x, y)$



ある三角形要素上で
$u(x,y)=$
$u^1A^1(x,y)+u^2A^2(x,y)+u^3A^3(x,y)$
とした1次関数は
$=ax+by+c$ とおけるので、
3点それぞれで、

$=u^1=ax^1+by^1+c \quad (x,y)=(x^1,y^1)$

$=u^2=ax^2+by^3+c \quad (x,y)=(x^2,y^2)$

$=u^3=ax^3+by^3+c \quad (x,y)=(x^3,y^3)$

を満たす。

$\boldsymbol{U}=[u^1,u^2,u^3]^{\mathrm{T}},\boldsymbol{A}=[a,b,c]^{\mathrm{T}},$
$\boldsymbol{X}=[x^1,x^2,x^3]^{\mathrm{T}},\boldsymbol{Y}=[y^1,y^2,y^3]^{\mathrm{T}},$
$\mathbf{1}=[1,1,1]^{\mathrm{T}}$
とおけば、連立方程式
$\boldsymbol{U}=[\boldsymbol{X}\boldsymbol{Y}\mathbf{1}]\boldsymbol{A}$
が得られるので
クラメールの公式より
$\boldsymbol{A}=[|\boldsymbol{U}\boldsymbol{Y}\mathbf{1}|,|\boldsymbol{X}\boldsymbol{U}\mathbf{1}|,|\boldsymbol{X}\boldsymbol{Y}\boldsymbol{U}|]^{\mathrm{T}}/|\boldsymbol{X}\boldsymbol{Y}\mathbf{1}|$

連立方程式の係数を表す式
$[|\boldsymbol{UY1}|,|\boldsymbol{XU1}|,|\boldsymbol{XYU}|]^{\mathrm{T}}/|\boldsymbol{XY1}|$
の中で最も変数の多い行列式

$$|\boldsymbol{XYU}|=\begin{vmatrix}x^1 & y^1 & u^1\\ x^2 & y^2 & u^2\\ x^3 & y^3 & u^3\end{vmatrix}$$
$$=u^1\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}-u^2\begin{vmatrix}x^1 & y^1\\ x^3 & y^3\end{vmatrix}+u^3\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}$$
$$=u^1\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}+u^2\begin{vmatrix}x^3 & y^3\\ x^1 & y^1\end{vmatrix}+u^3\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}$$

この、$|\boldsymbol{XYU}|$の展開式をもとに1次関数の各係数を求める。

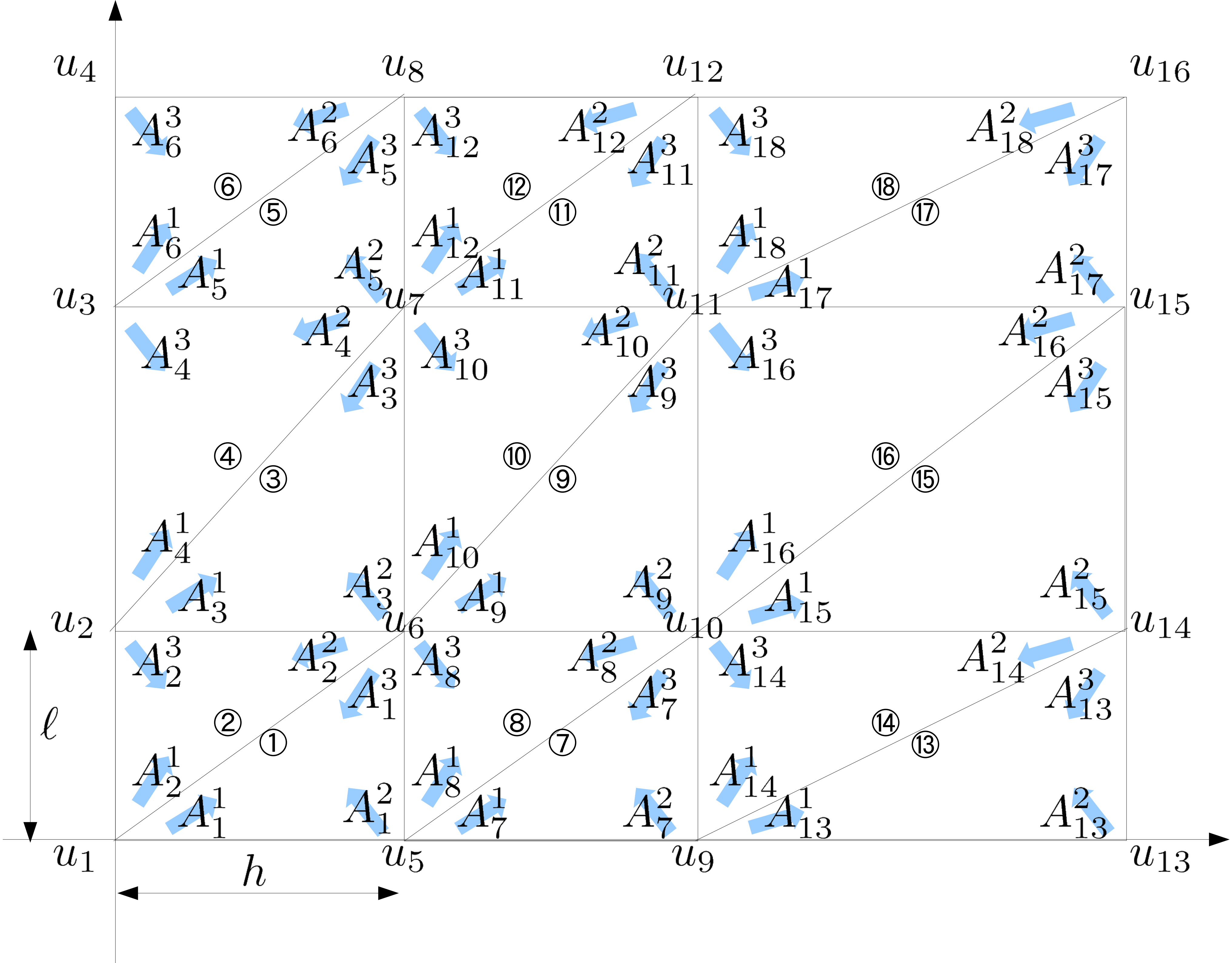
まず、分母に現れる$|\boldsymbol{XY\,1}|$は、$|\boldsymbol{XYU}|$の$\boldsymbol{U}$を$\mathbf{1}$に置き換えればよい

$$|\boldsymbol{XY1}|=\sum_l\begin{vmatrix}x^{l+1} & y^{l+1}\\ x^{l+2} & y^{l+2}\end{vmatrix}=\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}+\begin{vmatrix}x^3 & y^3\\ x^1 & y^1\end{vmatrix}+\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}$$

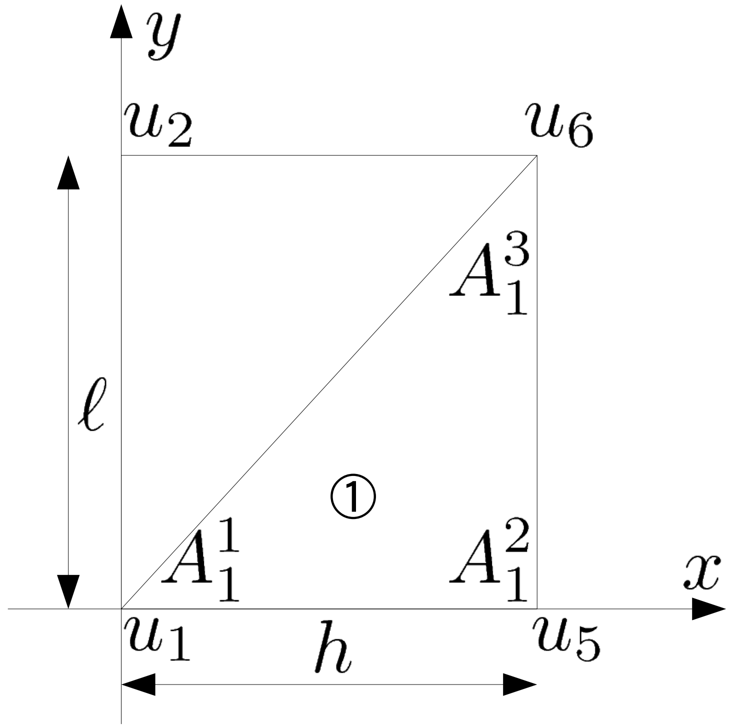
a,b の分子に現れる$|\boldsymbol{UY\,1}|$,$|\boldsymbol{XU\,1}|$は列の交換をしてみれば、$|\boldsymbol{XYU}|$の$\boldsymbol{X},\boldsymbol{Y}$を$\mathbf{1}$に置き換えて計算できることがわかる。

$$a=|\boldsymbol{UY1}|/|\boldsymbol{XY1}|$$
$$|\boldsymbol{UY1}|=-|\boldsymbol{1YU}|=-u^1\begin{vmatrix}1 & y^2\\ 1 & y^3\end{vmatrix}-u^2\begin{vmatrix}1 & y^3\\ 1 & y^1\end{vmatrix}-u^3\begin{vmatrix}1 & y^1\\ 1 & y^2\end{vmatrix}$$
$$b=|\boldsymbol{XU1}|/|\boldsymbol{XY1}|$$
$$|\boldsymbol{XU1}|=-|\boldsymbol{X1U}|=-u^1\begin{vmatrix}x^2 & 1\\ x^3 & 1\end{vmatrix}-u^2\begin{vmatrix}x^3 & 1\\ x^1 & 1\end{vmatrix}-u^3\begin{vmatrix}x^1 & 1\\ x^2 & 1\end{vmatrix}$$

$$\therefore\ ax+by+c=$$
$$=\left\{u^1\frac{\left[-\begin{vmatrix}1 & y^2\\ 1 & y^3\end{vmatrix}x-\begin{vmatrix}x^2 & 1\\ x^3 & 1\end{vmatrix}y+\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}\right]}{|\boldsymbol{XY1}|}+u^2\frac{\left[-\begin{vmatrix}1 & y^3\\ 1 & y^1\end{vmatrix}x-\begin{vmatrix}x^3 & 1\\ x^1 & 1\end{vmatrix}y+\begin{vmatrix}x^3 & y^3\\ x^1 & y^1\end{vmatrix}\right]}{|\boldsymbol{XY1}|}+u^3\frac{\left[-\begin{vmatrix}1 & y^1\\ 1 & y^2\end{vmatrix}x-\begin{vmatrix}x^1 & 1\\ x^2 & 1\end{vmatrix}y+\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}\right]}{|\boldsymbol{XY1}|}\right\}$$
$$=u^1\,A^1(x,y)+u^2\,A^2(x,y)+u^3\,A^3(x,y)$$



u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10} u_{11} u_{12} u_{13} u_{14} u_{15} u_{16}
h
ℓ
①②③④⑤⑥⑦⑧⑨⑩⑪⑫⑬⑭⑮⑯⑰⑱
A_1^1 A_1^2 A_1^3 A_2^1 A_2^2 A_2^3 A_3^1 A_3^2 A_3^3 A_4^1 A_4^2 A_4^3 A_5^1 A_5^2 A_5^3 A_6^1 A_6^2 A_6^3
A_7^1 A_7^2 A_7^3 A_8^1 A_8^2 A_8^3 A_9^1 A_9^2 A_9^3 A_{10}^1 A_{10}^2 A_{10}^3 A_{11}^1 A_{11}^2 A_{11}^3 A_{12}^1 A_{12}^2 A_{12}^3
A_{13}^1 A_{13}^2 A_{13}^3 A_{14}^1 A_{14}^2 A_{14}^3 A_{15}^1 A_{15}^2 A_{15}^3 A_{16}^1 A_{16}^2 A_{16}^3 A_{17}^1 A_{17}^2 A_{17}^3 A_{18}^1 A_{18}^2 A_{18}^3



$$\boldsymbol{X}=[x^1,x^2,x^3]^\mathrm{T}=[0,h,h]^\mathrm{T}, \boldsymbol{Y}=[y^1,y^2,y^3]^\mathrm{T}=[0,0,\ell]^\mathrm{T}$$

$$|\boldsymbol{XY1}|=\sum_l \begin{vmatrix} x^{l+1} & y^{l+1} \\ x^{l+2} & y^{l+2} \end{vmatrix} = \begin{vmatrix} x^2 & y^2 \\ x^3 & y^3 \end{vmatrix} + \begin{vmatrix} x^3 & y^3 \\ x^1 & y^1 \end{vmatrix} + \begin{vmatrix} x^1 & y^1 \\ x^2 & y^2 \end{vmatrix}$$

$$= \begin{vmatrix} h & 0 \\ h & \ell \end{vmatrix} + \begin{vmatrix} h & \ell \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ h & 0 \end{vmatrix} = h\ell = 2\times$$ ①の面積

$$u(x,y) = ax + by + c$$

$$=\left\{u^1\left[-\begin{vmatrix} 1 & y^2 \\ 1 & y^3 \end{vmatrix}x-\begin{vmatrix} x^2 & 1 \\ x^3 & 1 \end{vmatrix}y+\begin{vmatrix} x^2 & y^2 \\ x^3 & y^3 \end{vmatrix}\right]+u^2\left[-\begin{vmatrix} 1 & y^3 \\ 1 & y^1 \end{vmatrix}x-\begin{vmatrix} x^3 & 1 \\ x^1 & 1 \end{vmatrix}y+\begin{vmatrix} x^3 & y^3 \\ x^1 & y^1 \end{vmatrix}\right]+u^3\left[-\begin{vmatrix} 1 & y^1 \\ 1 & y^2 \end{vmatrix}x-\begin{vmatrix} x^1 & 1 \\ x^2 & 1 \end{vmatrix}y+\begin{vmatrix} x^1 & y^1 \\ x^2 & y^2 \end{vmatrix}\right]\right\}$$

$$/|\boldsymbol{XY1}|$$

$$=\left\{u^1\left[-\begin{vmatrix} 1 & 0 \\ 1 & \ell \end{vmatrix}x-\begin{vmatrix} h & 1 \\ h & 1 \end{vmatrix}y+\begin{vmatrix} h & \\ h & \ell \end{vmatrix}\right]+u^2\left[-\begin{vmatrix} 1 & \ell \\ 1 & 0 \end{vmatrix}x-\begin{vmatrix} h & 1 \\ 0 & 1 \end{vmatrix}y+\begin{vmatrix} h & \ell \\ 0 & 0 \end{vmatrix}\right]+u^3\left[-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}x-\begin{vmatrix} 0 & 1 \\ h & 1 \end{vmatrix}y+\begin{vmatrix} 0 & 0 \\ h & 0 \end{vmatrix}\right]\right\}/h\ell$$

$$= \{u^1[-\ell x - 0y + h\ell] + u^2[-\ell x - hy + 0] + u^3[0x + hy + 0]\}/[h\ell]$$

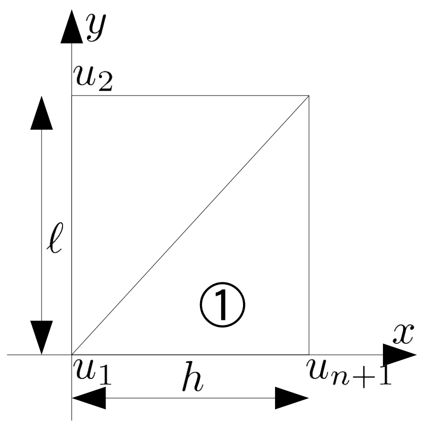
$$A^1(x,y) = -\frac{1}{h}x + 1, A^2(x,y) = -\frac{1}{h}x - \frac{1}{\ell}y, A^2(x,y) = -\frac{1}{\ell}y.$$

授業レポート用紙:氏名(　　　　　　　　　　)学籍番号(　　　　　　)

レポート(6)

図の有限要素①は$h\times l$の長方形要素を対角線で2分したものです。

- ①を台に持ち、1つの頂点で1、他の頂点で0の値をとる3つの1次関数を求めてください。



$\boldsymbol{X}=[x^1,x^2,x^3]^{\mathrm{T}}=[0,h,h]^{\mathrm{T}}, \boldsymbol{Y}=[y^1,y^2,y^3]^{\mathrm{T}}=[0,0,\ell]^{\mathrm{T}}$

$$|\boldsymbol{XY1}|=\sum_l\begin{vmatrix}x^{l+1} & y^{l+1}\\ x^{l+2} & y^{l+2}\end{vmatrix}=\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}+\begin{vmatrix}x^3 & y^3\\ x^1 & y^1\end{vmatrix}+\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}$$

$$=\begin{vmatrix}h & 0\\ h & \ell\end{vmatrix}+\begin{vmatrix}h & \ell\\ 0 & 0\end{vmatrix}+\begin{vmatrix}0 & 0\\ h & 0\end{vmatrix}=h\ell=2\times\text{①の面積}$$

任意の1次関数$ax+by=c$を①におき、3つの頂点でu_1,u_2,u_{n+1}の値をとることを考えれば、

$$u(x,y)=ax+by+c=\Bigg\{u^1\left[-\begin{vmatrix}1 & y^2\\ 1 & y^3\end{vmatrix}x-\begin{vmatrix}x^2 & 1\\ x^3 & 1\end{vmatrix}y+\begin{vmatrix}x^2 & y^2\\ x^3 & y^3\end{vmatrix}\right]$$

$$+u^2\left[-\begin{vmatrix}1 & y^3\\ 1 & y^1\end{vmatrix}x-\begin{vmatrix}x^3 & 1\\ x^1 & 1\end{vmatrix}y+\begin{vmatrix}x^3 & y^3\\ x^1 & y^1\end{vmatrix}\right]$$

$$+u^3\left[-\begin{vmatrix}1 & y^1\\ 1 & y^2\end{vmatrix}x-\begin{vmatrix}x^1 & 1\\ x^2 & 1\end{vmatrix}y+\begin{vmatrix}x^1 & y^1\\ x^2 & y^2\end{vmatrix}\right]\Bigg\}/|\boldsymbol{XY1}|$$

$$=\Bigg\{u^1\left[-\begin{vmatrix}1 & 0\\ 1 & \ell\end{vmatrix}x-\begin{vmatrix}h & 1\\ h & 1\end{vmatrix}y+\begin{vmatrix}h & \\ h & \ell\end{vmatrix}\right]+u^2\left[-\begin{vmatrix}1 & \ell\\ 1 & 0\end{vmatrix}x-\begin{vmatrix}h & 1\\ 0 & 1\end{vmatrix}y+\begin{vmatrix}h & \ell\\ 0 & 0\end{vmatrix}\right]$$

$$+u^3\left[-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}x-\begin{vmatrix}0 & 1\\ h & 1\end{vmatrix}y+\begin{vmatrix}0 & 0\\ h & 0\end{vmatrix}\right]\Bigg\}/h\ell$$

$$=\{u^1[-\ell x-0y+h\ell]+u^2[-\ell x-hy+0]$$

$$+u^3[0x+hy+0]\}/[h\ell]$$

と展開できることを用いて3つの1次関数を得る。

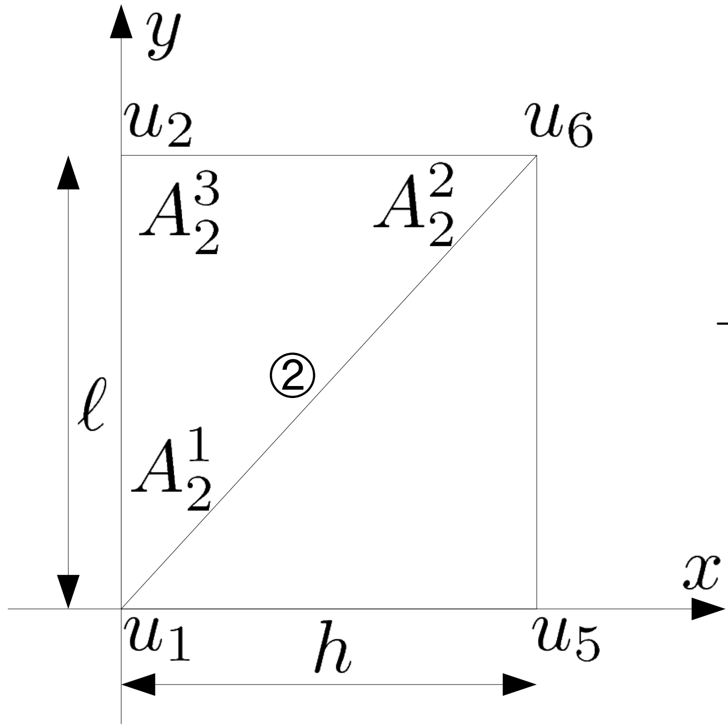
$$A^1(x,y)=-\frac{1}{h}x+1,$$

$$A^2(x,y)=-\frac{1}{h}x-\frac{1}{\ell}y,$$

$$A^2(x,y)=-\frac{1}{\ell}y.$$

2019年11月25日(月)

できれば授業の感想も書いてください。



$$\boldsymbol{X}=[x^1,x^2,x^3]^{\mathrm{T}}=[0,h,0]^{\mathrm{T}},\ \boldsymbol{Y}=[y^1,y^2,y^3]^{\mathrm{T}}=[0,\ell,\ell]^{\mathrm{T}}$$

$$u(x,y)=ax+by+c=\Big\{u^1\Big[-\begin{vmatrix}1&y^2\\1&y^3\end{vmatrix}x-\begin{vmatrix}x^2&1\\x^3&1\end{vmatrix}y+\begin{vmatrix}x^2&y^2\\x^3&y^3\end{vmatrix}\Big]$$

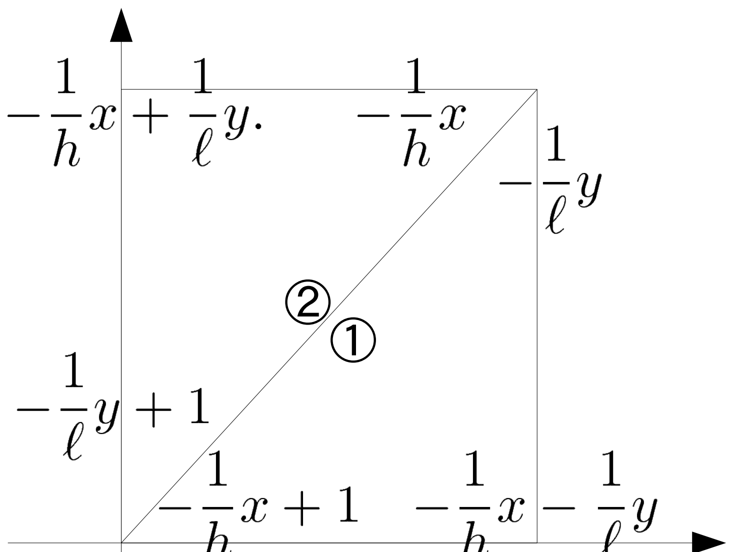
$$+u^2\Big[-\begin{vmatrix}1&y^3\\1&y^1\end{vmatrix}x-\begin{vmatrix}x^3&1\\x^1&1\end{vmatrix}y+\begin{vmatrix}x^3&y^3\\x^1&y^1\end{vmatrix}\Big]+u^3\Big[-\begin{vmatrix}1&y^1\\1&y^2\end{vmatrix}x-\begin{vmatrix}x^1&1\\x^2&1\end{vmatrix}y+\begin{vmatrix}x^1&y^1\\x^2&y^2\end{vmatrix}\Big]\Big\}$$

$$/|\boldsymbol{X}\boldsymbol{Y}\boldsymbol{1}|$$

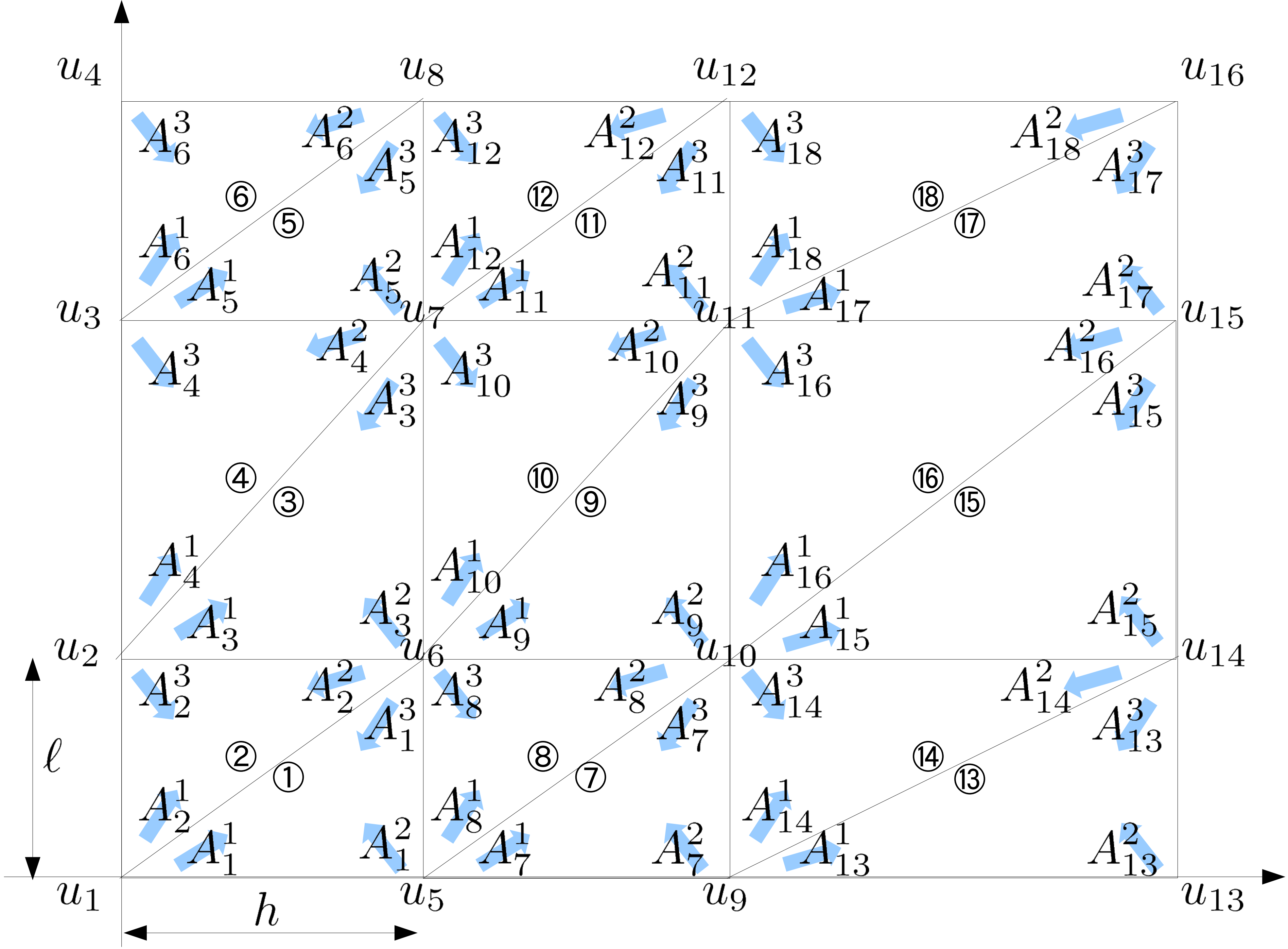
$$=\Big\{u^1\Big[-\begin{vmatrix}1&\ell\\1&\ell\end{vmatrix}x-\begin{vmatrix}h&1\\0&1\end{vmatrix}y+\begin{vmatrix}h&\ell\\0&\ell\end{vmatrix}\Big]+u^2\Big[-\begin{vmatrix}1&\ell\\1&0\end{vmatrix}x-\begin{vmatrix}0&1\\0&1\end{vmatrix}y+\begin{vmatrix}0&\ell\\0&0\end{vmatrix}\Big]$$

$$+u^3\Big[-\begin{vmatrix}1&0\\1&\ell\end{vmatrix}x-\begin{vmatrix}0&1\\h&1\end{vmatrix}y+\begin{vmatrix}0&0\\h&\ell\end{vmatrix}\Big]\Big\}/h\ell$$

$$=\{u^1[0x-hy+h\ell]+u^2[+\ell x-0y+0]+u^3[-\ell x+hy+0]\}/[h\ell]$$



その他の三角形要素も、①・②を平行移動すれば同じ



1) y軸優先で頂点u_1~u_{16}を定める

2) 頂点番号順に有限要素①〜⑱を定める

3) 有限要素毎に頂点番号が一番若いものから左回りにA^1~A^3を定める

連立方程式の導出

- 例:Laplace方程式の境界値問題

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=0 \qquad 0<x,y<1,$$

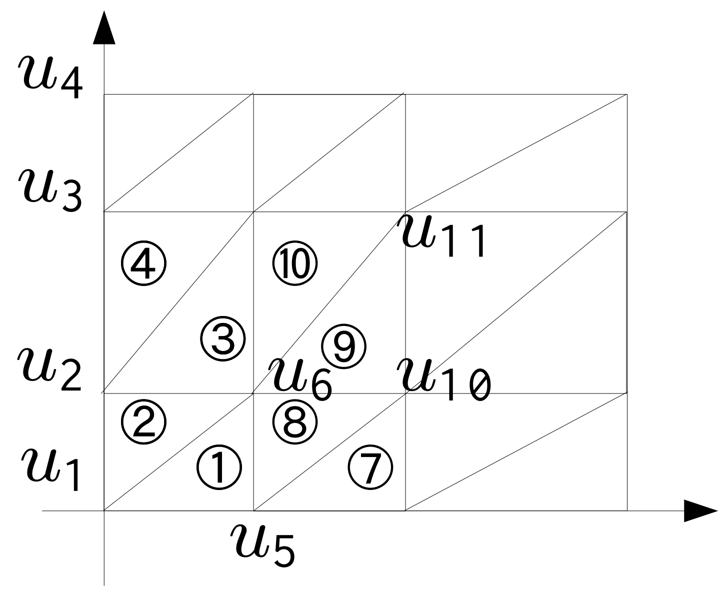
$$u(x,0)=0,\ u(x,1)=1,\ u(0,y)\equiv(1,y)=y.$$

離散化:有限要素を定める
それぞれに1次関数を定める

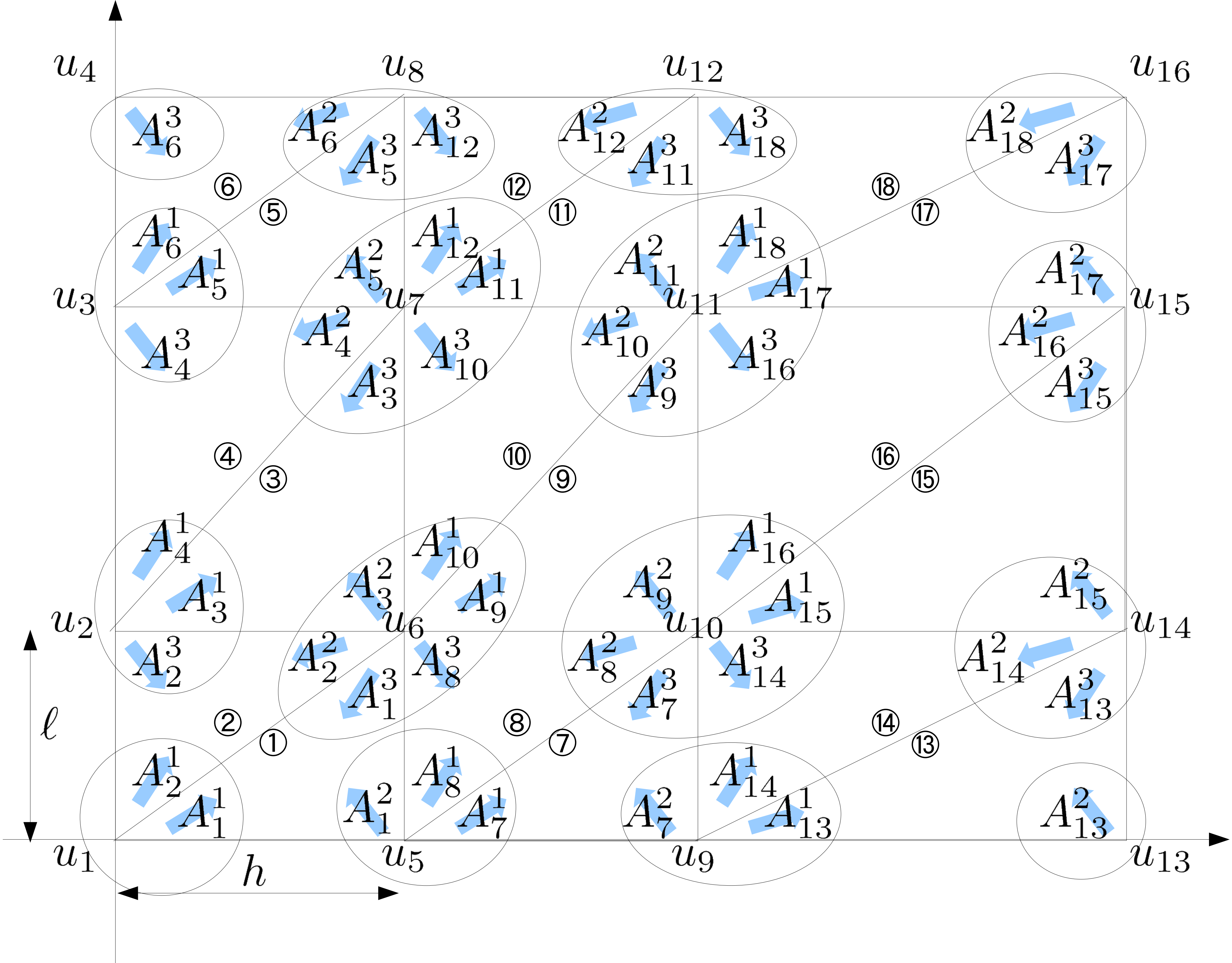
$\sum_j a_j x + b_j y + c_j$

$\Rightarrow \sum_j \sum_k u_j^k A_j^k(x,y)$

u_j^kは端点値:全て独立ではない



隣接する有限要素の境界で関数は連続
→例えば u_6 は①②③⑧⑨⑩に接する



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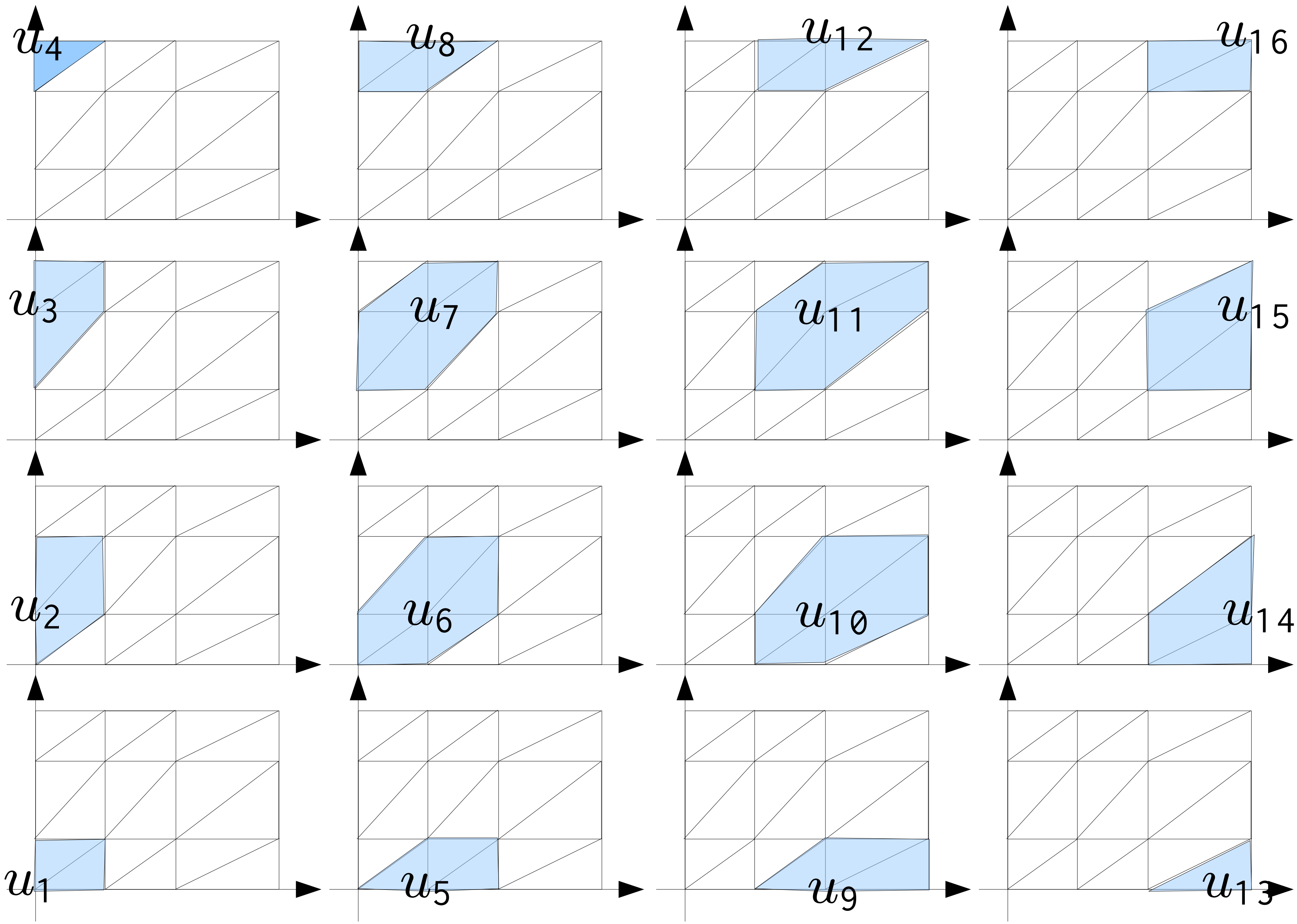
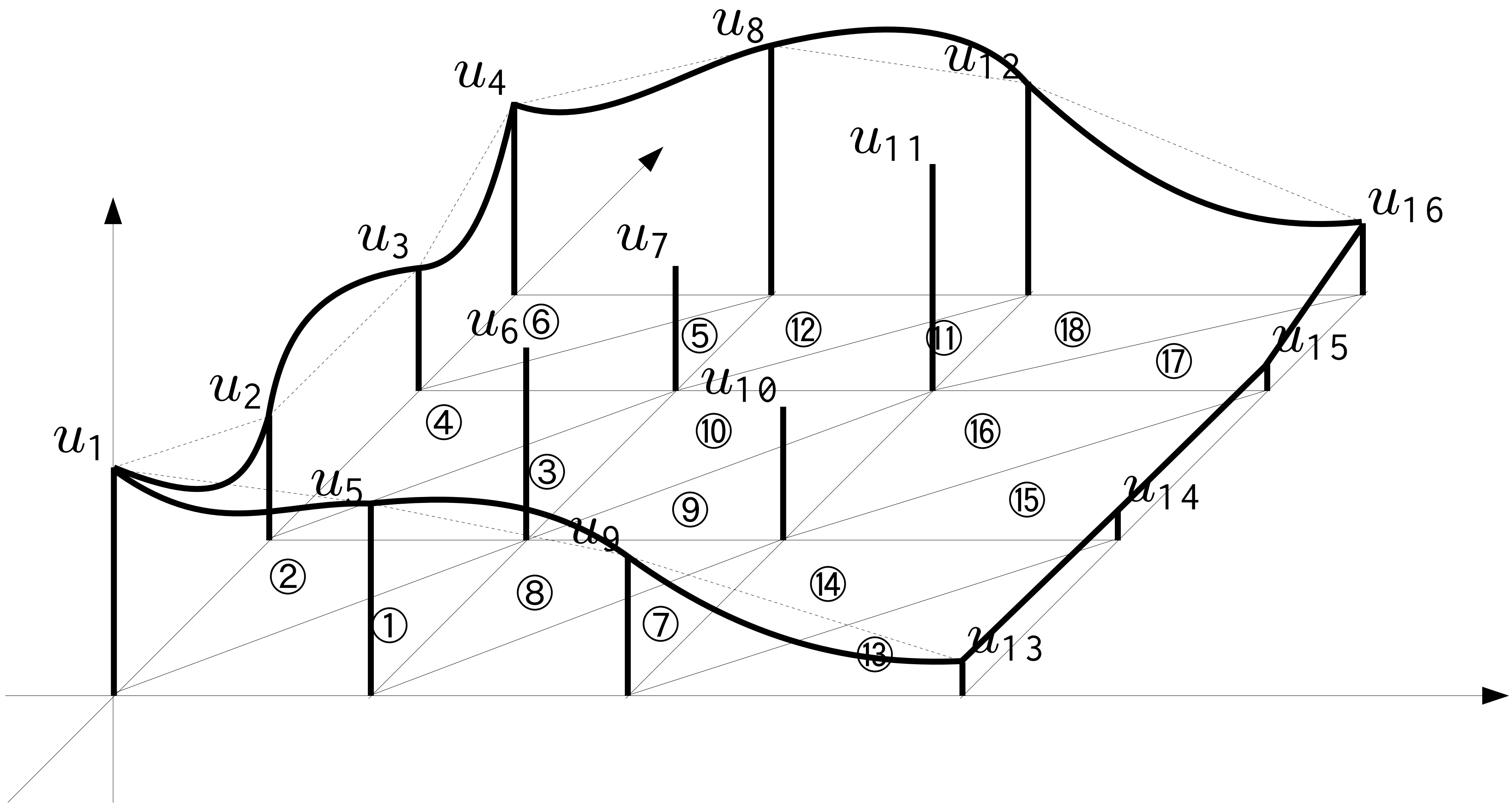
A_{13}^2

連立方程式の導出

- 変数毎の台



連立方程式の導出



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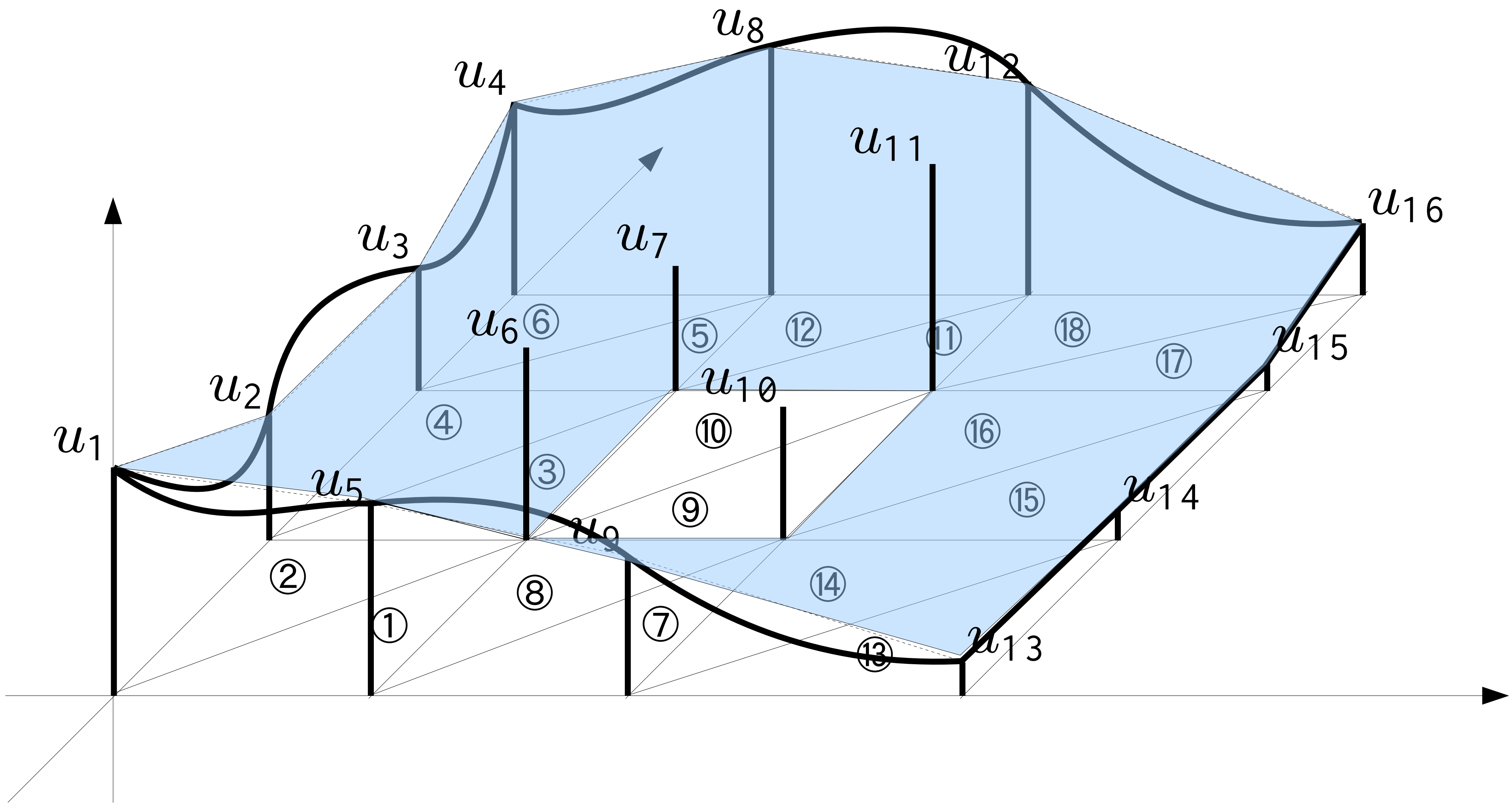
連立方程式の導出



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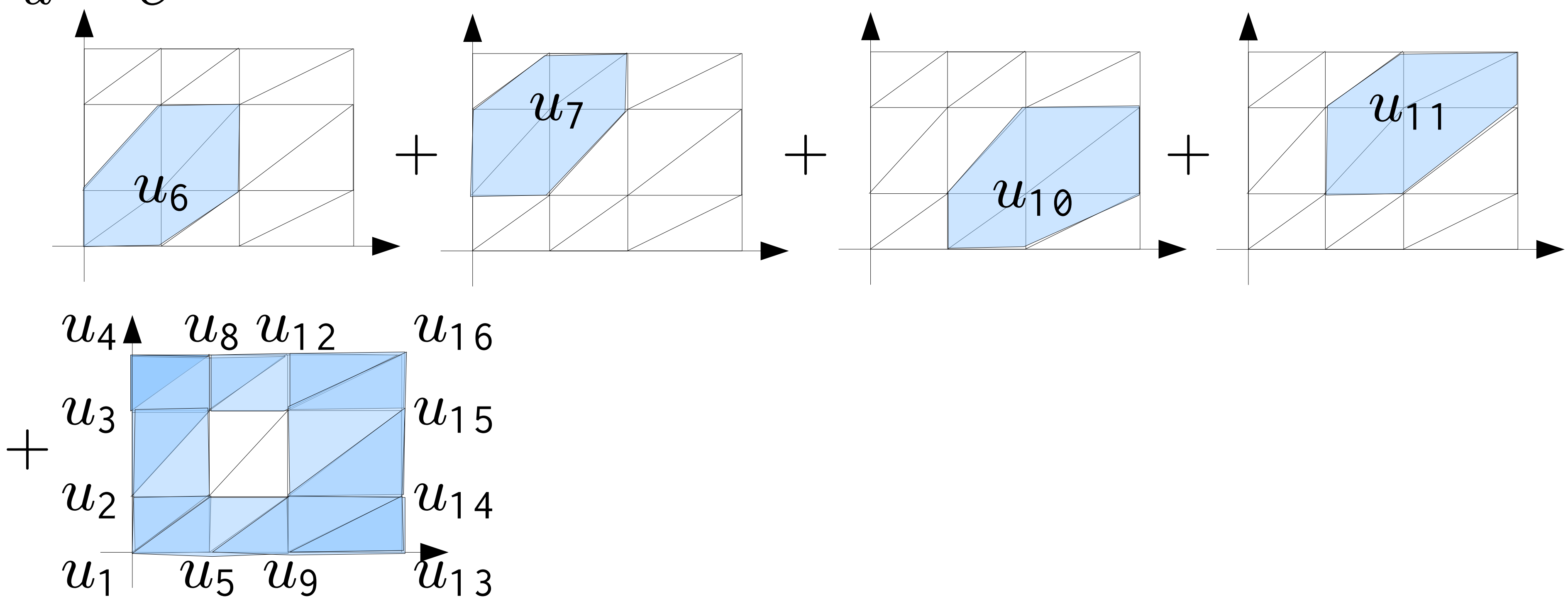
連立方程式の導出

- 独立変数と定数

$u \sim U =$



- u_6, u_7, u_{10}, u_{11}だけを未定係数だと思えば良い
⇒それぞれ6つの有限要素を台とする4つの基底関数を使ったGalerkin法と考える

連立方程式の導出

- 定数部分 $\sum_{j=1,2,3,4,5,8,9,12,13,14,15,16} u_j \phi_j(x,y)$

$\phi_1 = A_1^1 + A_2^1$

$\phi_2 = A_2^3 + A_3^1 + A_4^1$

$\phi_3 = A_4^3 + A_5^1 + A_6^1$

$\phi_4 = A_6^3$

$\phi_5 = A_1^2 + A_7^1 + A_8^1$

$\phi_8 = A_5^3 + A_6^2 + A_{12}^3$

$\phi_9 = A_7^2 + A_{13}^1 + A_{14}^1$

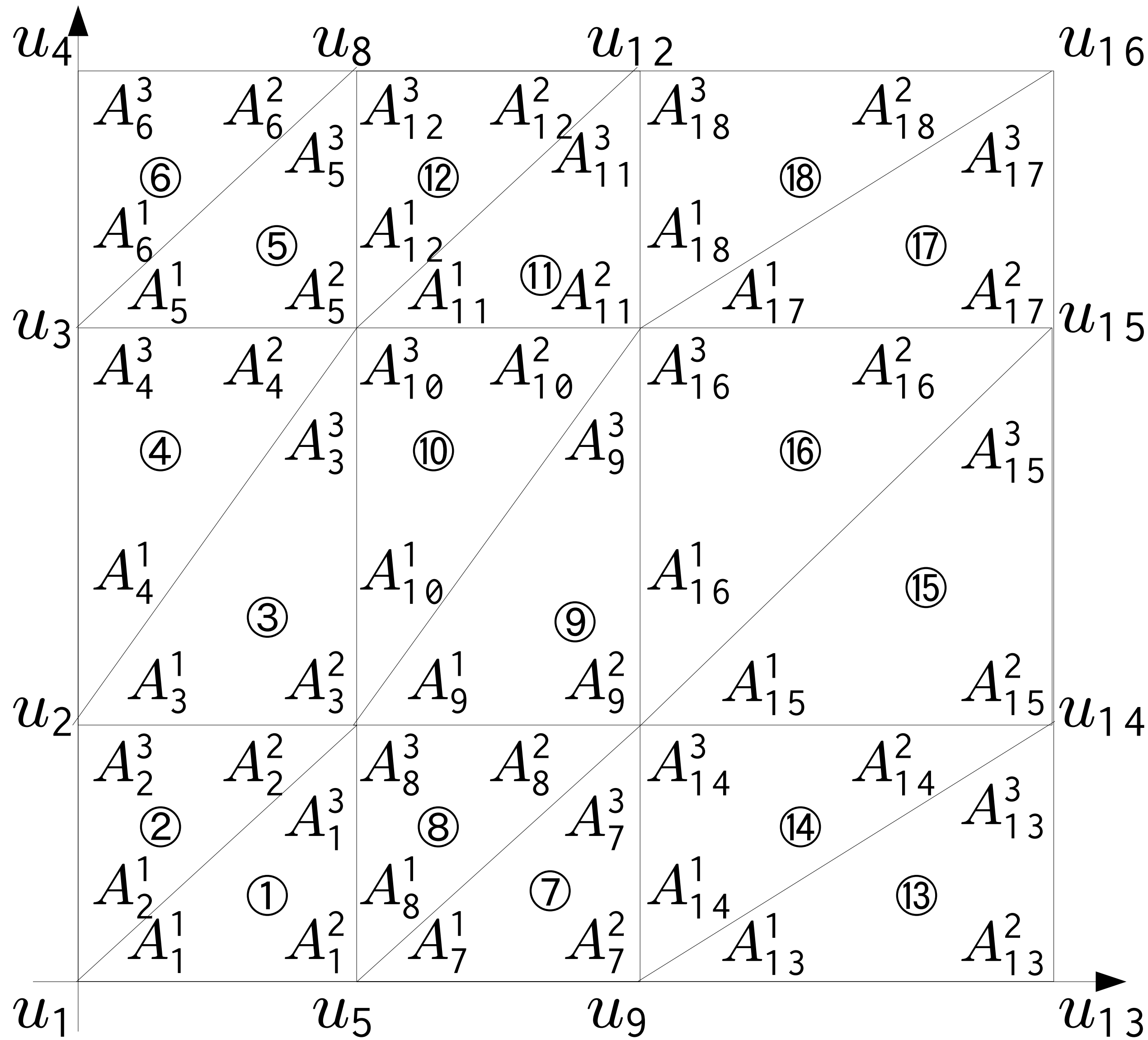
$\phi_{12} = A_{11}^3 + A_{12}^2 + A_{18}^3$

$\phi_{13} = A_{13}^2$

$\phi_{14} = A_{13}^3 + A_{14}^2 + A_{15}^2$

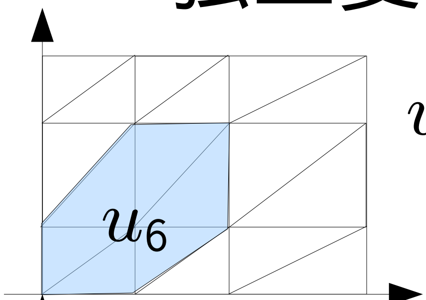
$\phi_{15} = A_{15}^3 + A_{16}^2 + A_{17}^2$

$\phi_{16} = A_{17}^3 + A_{18}^2$

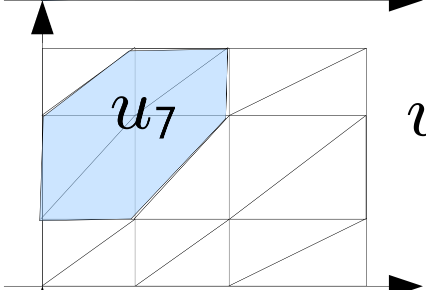


連立方程式の導出

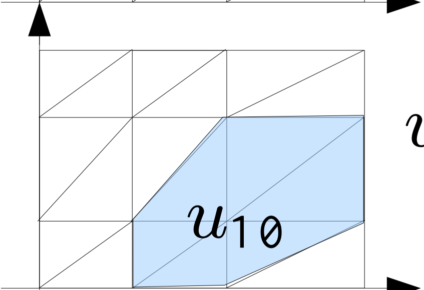
- 独立変数と基底関数



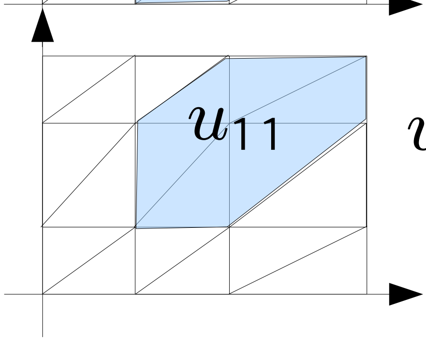
$$u_6\varphi_6(x,y) = u_6[A_1^3(x,y) + A_2^2(x,y) + A_3^2(x,y) + A_8^3(x,y) + A_9^1(x,y) + A_{10}^1(x,y)]$$



$$u_7\varphi_7(x,y) = u_7[A_3^3(x,y) + A_4^2(x,y) + A_5^2(x,y) + A_{10}^3(x,y) + A_{11}^1(x,y) + A_{12}^1(x,y)]$$



$$u_{10}\varphi_{10}(x,y) = u_{10}[A_7^3(x,y) + A_8^2(x,y) + A_9^2(x,y) + A_{14}^3(x,y) + A_{15}^1(x,y) + A_{16}^1(x,y)]$$



$$u_{11}\varphi_{11}(x,y) = u_{11}[A_9^3(x,y) + A_{10}^2(x,y) + A_{11}^2(x,y) + A_{16}^3(x,y) + A_{17}^1(x,y) + A_{18}^1(x,y)]$$

$$u \sim U = \sum_{j=1,2,3,4,5,8,9,12,13,14,15,16} u_j \phi_j(x,y) + \sum_{j=6,7,10,11} u_j \varphi_j(x,y)$$

$$= u_1[A_1^1 + A_2^1] + u_2[A_2^3 + A_3^1 + A_4^1] + u_3[A_4^3 + A_5^1 + A_6^1] + u_4[A_6^3]$$

$$+ u_5[A_1^2 + A_7^1 + A_8^1] + u_6 \underbrace{[A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1]}_{(= \varphi_6)}$$

$$+ u_7 \underbrace{[A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1]}_{(= \varphi_7)} + u_8[A_5^3 + A_6^2 + A_{12}^3]$$

$$+ u_9[A_7^2 + A_{13}^1 + A_{14}^1] + u_{10} \underbrace{[A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1]}_{(= \varphi_{10})}$$

$$+ u_{11} \underbrace{[A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1]}_{(= \varphi_{11})} + u_{12}[A_{11}^3 + A_{12}^2 + A_{18}^3]$$

$$+ u_{13}[A_{13}^2] + u_{14}[A_{13}^3 + A_{14}^2 + A_{15}^2]$$

$$+ u_{15}[A_{15}^3 + A_{16}^2 + A_{17}^2] + u_{16}[A_{17}^3 + A_{18}^2]$$

残差方程式と弱形式

- Laplace方程式の重み付き残差

$$\iint \varphi_k \triangle \Big(\sum_{j\neq 6,7,10,11} u_j \phi_j + \sum_{j=6,7,10,11} u_j \varphi_j \Big) \mathrm{d}x\mathrm{d}y = 0$$

- Laplace方程式の重み付き残差（弱形式）

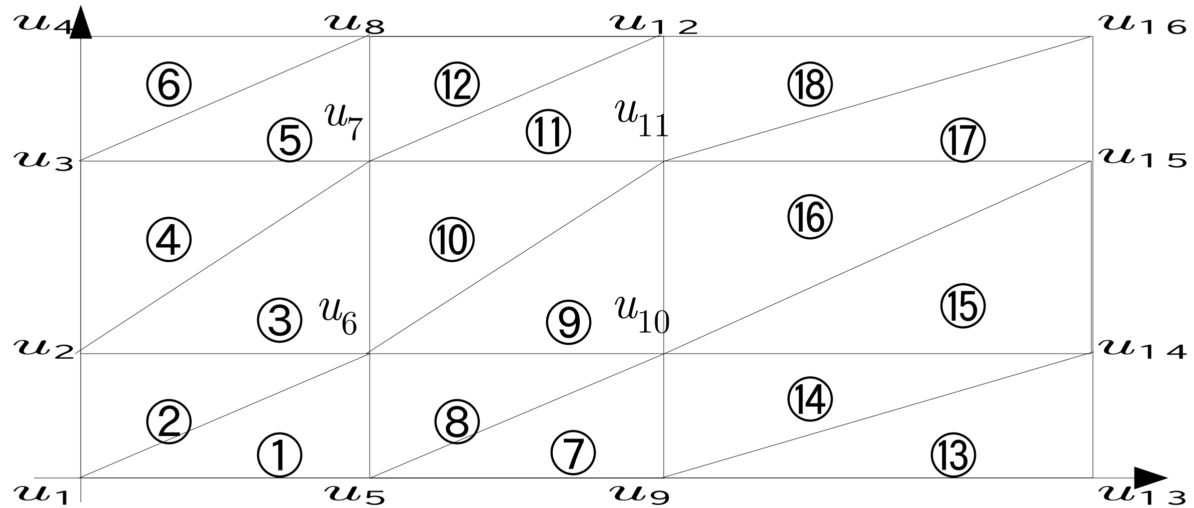
$$\sum_{j\neq 6,7,10,11} u_j \iint \frac{\partial \varphi_k}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \varphi_k}{\partial y}\frac{\partial \phi_j}{\partial y} \mathrm{d}x\mathrm{d}y + \sum_{j=6,7,10,11} u_j \iint \frac{\partial \varphi_k}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_k}{\partial y}\frac{\partial \varphi_j}{\partial y} \mathrm{d}x\mathrm{d}y = 0$$

- ϕ と φ は ${A_p}^s, {A_q}^t$ の組合せ（p,q有限要素の番号、s,t要素毎の頂点番号）なので積分計算も有限要素毎に考える。

$$\iint \frac{\partial A_j^l}{\partial x}\frac{\partial A_k^m}{\partial x} + \frac{\partial A_j^l}{\partial y}\frac{\partial A_k^m}{\partial y} \mathrm{d}x\mathrm{d}y = \delta_{jk}[a_j^l a_k^m + b_j^l b_k^m] S_j$$

δ_{jk} はクロネッカデルタ、S_j は有限要素 j の面積

- 変数/定数
  毎にまとめる



$$u \sim U = u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + u_4\phi_4$$
$$+u_5\phi_5 + u_6\varphi_6 + u_7\varphi_7 + u_8\phi_8 + u_9\phi_9$$
$$+u_{10}\varphi_{10} + u_{11}\varphi_{11} + u_{12}\phi_{12} + u_{13}\phi_{13} +$$
$$u_{14}\phi_{14} + u_{15}\phi_{15} + u_{16}\phi_{16} + u_{17}\phi_{17} + u_{18}\phi_{18}$$

$$\phi_1 = A_1^1 + A_2^1$$
$$\phi_2 = A_2^3 + A_3^1 + A_4^1$$
$$\phi_3 = A_4^3 + A_5^1 + A_6^1$$
$$\phi_4 = A_6^3$$
$$\phi_5 = A_1^2 + A_7^1 + A_8^1$$
$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1$$
$$\varphi_7 = A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1$$
$$\phi_8 = A_5^3 + A_6^2 + A_{12}^3$$

$$\phi_9 = A_7^2 + A_{13}^1 + A_{14}^1$$
$$\varphi_{10} = A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1$$
$$\varphi_{11} = A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1$$
$$\phi_{12} = A_{11}^3 + A_{12}^2 + A_{18}^3$$
$$\phi_{13} = A_{13}^2$$
$$\phi_{14} = A_{13}^3 + A_{14}^2 + A_{15}^2$$
$$\phi_{15} = A_{15}^3 + A_{16}^2 + A_{17}^2$$
$$\phi_{16} = A_{17}^3 + A_{18}^2$$

- 残差の弱形式を基底関数毎にゼロにする

$$
\begin{aligned}
u \sim U = & u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + u_4\phi_4 \\
& + u_5\phi_5 + u_6\varphi_6 + u_7\varphi_7 + u_8\phi_8 + u_9\phi_9 \\
& + u_{10}\varphi_{10} + u_{11}\varphi_{11} + u_{12}\phi_{12} + u_{13}\phi_{13} + \\
& u_{14}\phi_{14} + u_{15}\phi_{15} + u_{16}\phi_{16} + u_{17}\phi_{17} + u_{18}\phi_{18}
\end{aligned}
$$

$$\int\int \frac{\partial \varphi_k}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial \varphi_k}{\partial y}\frac{\partial U}{\partial y}\mathrm{d}x\mathrm{d}y = 0$$

Uを基底関数に分けて考えれば

$$\sum_j u_j \int\int \frac{\partial \varphi_k}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \varphi_k}{\partial y}\frac{\partial \phi_j}{\partial y}\mathrm{d}x\mathrm{d}y = 0 \quad k = 6, 7, 10, 11, (\phi_j = \varphi_j \ j = 6, 7, 10, 11)$$

φ_kとϕ_j,φ_j がどちらもゼロでない有限要素上の計算を考えれば良い
それが有限要素 i であるなら

$$\int\int \frac{\partial A_i^p}{\partial x}\frac{\partial A_i^q}{\partial x} + \frac{\partial A_i^p}{\partial y}\frac{\partial A_i^q}{\partial y}\mathrm{d}x\mathrm{d}y = a_i^p a_i^q + b_i^p b_i^q \equiv B_i^{pq}$$

を計算することになる

$$u \sim U = \sum_{j=6,7,10,11} u_j \varphi_j + \sum_{j \neq 6,7,10,11} u_j \phi_j$$

$$\begin{aligned}
\phi_1 &= \boxed{A_1^1} + \boxed{A_2^1} \\
\phi_2 &= \boxed{A_2^3} + \boxed{A_3^1} + A_4^1 \\
\phi_3 &= A_4^3 + A_5^1 + A_6^1 \\
\phi_4 &= A_6^3 \\
\phi_5 &= \boxed{A_1^2} + A_7^1 + \boxed{A_8^1} \\
\varphi_6 &= \boxed{A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1} \\
\varphi_7 &= \boxed{A_3^3} + A_4^2 + A_5^2 + \boxed{A_{10}^3} + A_{11}^1 + A_{12}^1 \\
\phi_8 &= A_5^3 + A_6^2 + A_{12}^3
\end{aligned}$$

$$\begin{aligned}
\phi_9 &= A_7^2 + A_{13}^1 + A_{14}^1 \\
\varphi_{10} &= A_7^3 + \boxed{A_8^2} + \boxed{A_9^2} + A_{14}^3 + A_{15}^1 + A_{16}^1 \\
\varphi_{11} &= \boxed{A_9^3} + \boxed{A_{10}^2} + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1 \\
\phi_{12} &= A_{11}^3 + A_{12}^2 + A_{18}^3 \\
\phi_{13} &= A_{13}^2 \\
\phi_{14} &= A_{13}^3 + A_{14}^2 + A_{15}^2 \\
\phi_{15} &= A_{15}^3 + A_{16}^2 + A_{17}^2 \\
\phi_{16} &= A_{17}^3 + A_{18}^2
\end{aligned}$$

$$\begin{aligned}
\varphi_6 &= \boxed{A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1} \\
\varphi_7 &= A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1
\end{aligned}$$

$$\begin{aligned}
\varphi_{10} &= A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1 \\
\varphi_{11} &= A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1
\end{aligned}$$

$$\begin{aligned}
&\sum_k u_j \iint \frac{\partial \varphi_6}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \varphi_6}{\partial y}\frac{\partial \phi_j}{\partial y}\,\mathrm{d}x\mathrm{d}y \\
&= u_1(B_1^{31} + B_2^{12}) + u_2(B_2^{23} + B_3^{12}) + u_5(B_1^{23} + B_8^{31}) \\
&\quad + u_6(B_1^{33} + B_2^{22} + B_3^{22} + B_8^{33} + B_9^{11} + B_{10}^{11}) + u_7(B_3^{23} + B_{10}^{31}) \\
&\quad + u_{10}(B_8^{23} + B_9^{12}) + u_{11}(B_9^{31} + B_{10}^{12}) = 0
\end{aligned}$$

$$u \sim U = \sum_{j=6,7,10,11} u_j \varphi_j + \sum_{j \neq 6,7,10,11} u_j \phi_j$$

$$\phi_1 = A_1^1 + A_2^1$$
$$\phi_2 = A_2^3 + \boxed{A_3^1} + \boxed{A_4^1}$$
$$\phi_3 = \boxed{A_4^3} + \boxed{A_5^1} + A_6^1$$
$$\phi_4 = A_6^3$$
$$\phi_5 = A_1^2 + A_7^1 + A_8^1$$
$$\varphi_6 = A_1^3 + A_2^2 + \boxed{A_3^2} + A_8^3 + A_9^1 + \boxed{A_{10}^1}$$
$$\varphi_7 = \boxed{A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1}$$
$$\phi_8 = \boxed{A_5^3} + A_6^2 + \boxed{A_{12}^3}$$

$$\phi_9 = A_7^2 + A_{13}^1 + A_{14}^1$$
$$\varphi_{10} = A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1$$
$$\varphi_{11} = A_9^3 + \boxed{A_{10}^2} + \boxed{A_{11}^2} + A_{16}^3 + A_{17}^1 + A_{18}^1$$
$$\phi_{12} = \boxed{A_{11}^3} + \boxed{A_{12}^2} + A_{18}^3$$
$$\phi_{13} = A_{13}^2$$
$$\phi_{14} = A_{13}^3 + A_{14}^2 + A_{15}^2$$
$$\phi_{15} = A_{15}^3 + A_{16}^2 + A_{17}^2$$
$$\phi_{16} = A_{17}^3 + A_{18}^2$$

$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1$$
$$\varphi_7 = \boxed{A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1}$$
$$\varphi_{10} = A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1$$
$$\varphi_{11} = A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1$$

$$\sum_{\mathsf{k}} u_{\mathsf{k}} \iint \frac{\partial \varphi_7}{\partial x}\frac{\partial \phi_{\mathsf{k}}}{\partial x} + \frac{\partial \varphi_7}{\partial y}\frac{\partial \phi_{\mathsf{k}}}{\partial y}\,\mathrm{d}x\mathrm{d}y$$
$$= u_2(B_3^{31} + B_4^{12}) + u_3(B_4^{23} + B_5^{12}) + u_6(B_3^{23} + B_{10}^{31})$$
$$+ u_7(B_3^{33} + B_4^{22} + B_5^{22} + B_{10}^{33} + B_{11}^{22} + B_{12}^{11}) + u_8(B_5^{23} + B_{12}^{31}) + u_{11}(B_{10}^{23} + B_{11}^{12}) + u_{12}(B_{11}^{31} + B_{12}^{12})$$

$$u \sim U = \sum_{j=6,7,10,11} u_j \varphi_j + \sum_{j\neq 6,7,10,11} u_j \phi_j$$

$$\phi_1 = A_1^1 + A_2^1$$
$$\phi_2 = A_2^3 + A_3^1 + A_4^1$$
$$\phi_3 = A_4^3 + A_5^1 + A_6^1$$
$$\phi_4 = A_6^3$$
$$\phi_5 = A_1^2 + \boxed{A_7^1} + \boxed{A_8^1}$$
$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + \boxed{A_8^3} + \boxed{A_9^1} + A_{10}^1$$
$$\varphi_7 = A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1$$
$$\phi_8 = A_5^3 + A_6^2 + A_{12}^3$$

$$\phi_9 = \boxed{A_7^2} + A_{13}^1 + \boxed{A_{14}^1}$$
$$\varphi_{10} = \boxed{A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1}$$
$$\varphi_{11} = \boxed{A_9^3} + A_{10}^2 + A_{11}^2 + \boxed{A_{16}^3} + A_{17}^1 + A_{18}^1$$
$$\phi_{12} = A_{11}^3 + A_{12}^2 + A_{18}^3$$
$$\phi_{13} = A_{13}^2$$
$$\phi_{14} = A_{13}^3 + \boxed{A_{14}^2} + \boxed{A_{15}^2}$$
$$\phi_{15} = \boxed{A_{15}^3} + \boxed{A_{16}^2} + A_{17}^2$$
$$\phi_{16} = A_{17}^3 + A_{18}^2$$

$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1 \qquad \varphi_{10} = \boxed{A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1}$$
$$\varphi_7 = A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1 \qquad \varphi_{11} = A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1$$

$$\sum_k u_k \iint \frac{\partial \varphi_{10}}{\partial x}\frac{\partial \phi_k}{\partial x} + \frac{\partial \varphi_{10}}{\partial y}\frac{\partial \phi_k}{\partial y} \mathrm{d}x\mathrm{d}y$$
$$= u_5(B_7^{31} + B_8^{12}) + u_6(B_8^{23} + B_9^{12}) + u_9(B_7^{23} + B_{14}^{31})$$
$$+ u_{10}(B_7^{33} + B_8^{22} + B_9^{22} + B_{14}^{33} + B_{15}^{11} + B_{16}^{11})$$
$$+ u_{11}(B_9^{23} + B_{16}^{31}) + u_{14}(B_{14}^{23} + B_{15}^{12}) + u_{15}(B_{15}^{23} + B_{16}^{12}) = 0$$

$$u \sim U = \sum_{j=6,7,10,11} u_j \varphi_j + \sum_{j\neq 6,7,10,11} u_j \phi_j$$

$$\phi_1 = A_1^1 + A_2^1 \qquad \phi_9 = A_7^2 + A_{13}^1 + A_{14}^1$$

$$\phi_2 = A_2^3 + A_3^1 + A_4^1 \qquad \varphi_{10} = A_7^3 + A_8^2 + \boxed{A_9^2} + A_{14}^3 + A_{15}^1 + \boxed{A_{16}^1}$$

$$\phi_3 = A_4^3 + A_5^1 + A_6^1 \qquad \varphi_{11} = \boxed{A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1}$$

$$\phi_4 = A_6^3 \qquad \phi_{12} = \boxed{A_{11}^3} + A_{12}^2 + \boxed{A_{18}^3}$$

$$\phi_5 = A_1^2 + A_7^1 + A_8^1 \qquad \phi_{13} = A_{13}^2$$

$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + A_8^3 + \boxed{A_9^1} + \boxed{A_{10}^1} \qquad \phi_{14} = A_{13}^3 + A_{14}^2 + A_{15}^2$$

$$\varphi_7 = A_3^3 + A_4^2 + A_5^2 + \boxed{A_{10}^3} + \boxed{A_{11}^1} + A_{12}^1 \qquad \phi_{15} = A_{15}^3 + \boxed{A_{16}^2} + \boxed{A_{17}^2}$$

$$\phi_8 = A_5^3 + A_6^2 + A_{12}^3 \qquad \phi_{16} = \boxed{A_{17}^3} + \boxed{A_{18}^2}$$

$$\varphi_6 = A_1^3 + A_2^2 + A_3^2 + A_8^3 + A_9^1 + A_{10}^1 \qquad \varphi_{10} = A_7^3 + A_8^2 + A_9^2 + A_{14}^3 + A_{15}^1 + A_{16}^1$$

$$\varphi_7 = A_3^3 + A_4^2 + A_5^2 + A_{10}^3 + A_{11}^1 + A_{12}^1 \qquad \varphi_{11} = \boxed{A_9^3 + A_{10}^2 + A_{11}^2 + A_{16}^3 + A_{17}^1 + A_{18}^1}$$

$$\sum_k u_k \iint \frac{\partial \varphi_{11}}{\partial x}\frac{\partial \phi_k}{\partial x} + \frac{\partial \varphi_{11}}{\partial y}\frac{\partial \phi_k}{\partial y}\,\mathrm{d}x\mathrm{d}y$$

$$= u_6\left(B_9^{31} + B_{10}^{12}\right) + u_7\left(B_{10}^{23} + B_{11}^{12}\right) + u_{10}\left(B_7^{23} + B_{16}^{31}\right)$$

$$+u_{11}(B_9^{33} + B_{10}^{22} + B_{11}^{22} + B_{16}^{33} + B_{17}^{11} + B_{18}^{11})$$

$$+u_{14}(B_{11}^{23} + B_{18}^{31}) + u_{15}(B_{16}^{23} + B_{16}^{12}) + u_{16}(B_{17}^{23} + B_{18}^{12}) = 0$$

$$u_1(B_1^{31}+B_2^{12})+u_2(B_2^{23}+B_3^{12})+u_5(B_1^{23}+B_8^{31})$$
$$+\boxed{u_6}(B_1^{33}+B_2^{22}+B_3^{22}+B_8^{33}+B_9^{11}+B_{10}^{11})+\boxed{u_7}(B_3^{23}+B_{10}^{31})$$
$$+\boxed{u_{10}}(B_8^{23}+B_9^{12})+\boxed{u_{11}}(B_9^{31}+B_{10}^{12})=0$$

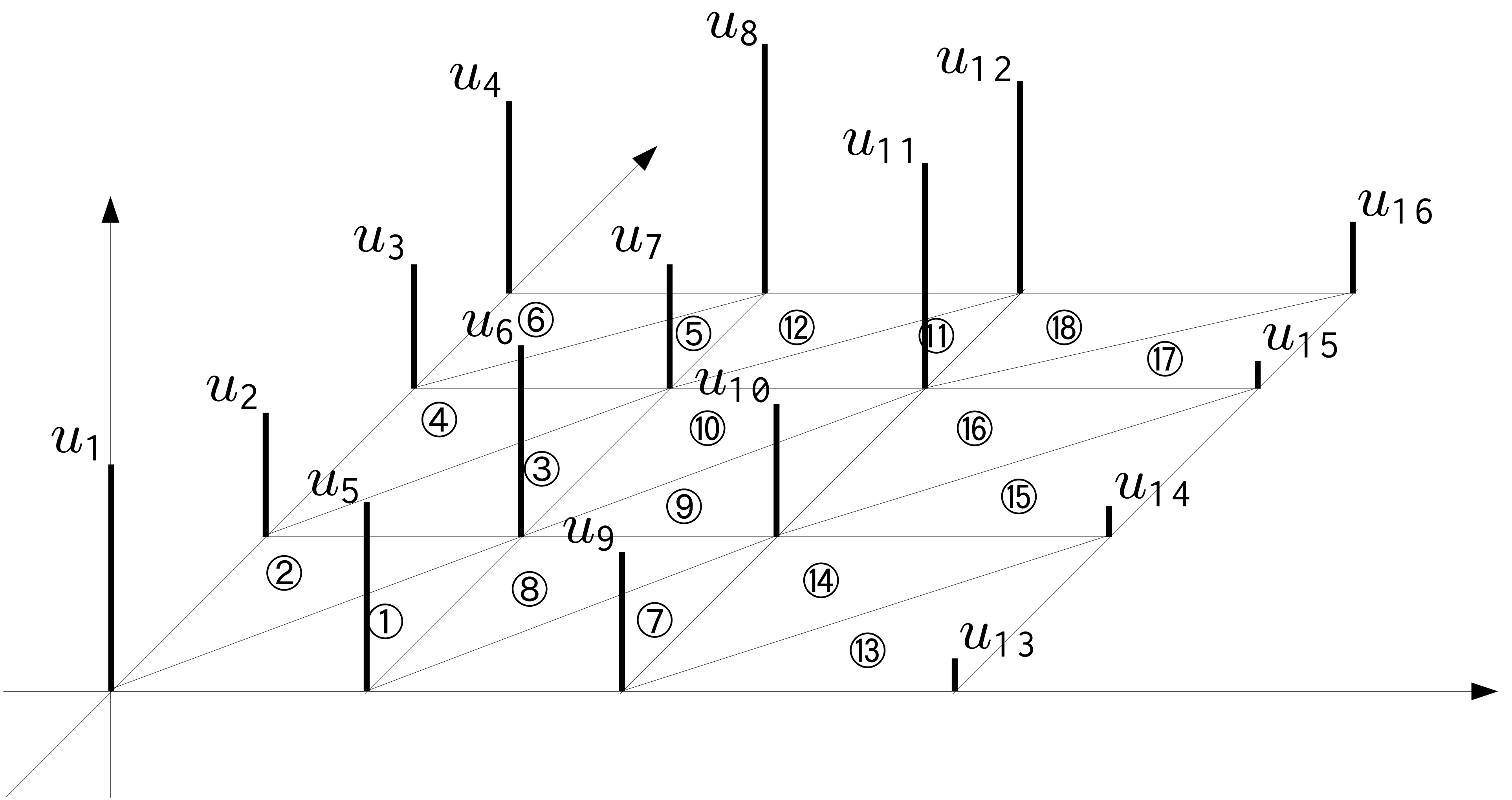
$$u_2(B_3^{31}+B_4^{12})+u_3(B_4^{23}+B_5^{12})+\boxed{u_6}(B_3^{23}+B_{10}^{31})$$
$$+\boxed{u_7}(B_3^{33}+B_4^{22}+B_5^{22}+B_{10}^{33}+B_{11}^{22}+B_{12}^{11})+u_8(B_5^{23}+B_{12}^{31})$$
$$+\boxed{u_{11}}(B_{10}^{23}+B_{11}^{12})+u_{12}(B_{11}^{31}+B_{12}^{12})=0$$

$$u_5(B_7^{31}+B_8^{12})+\boxed{u_6}(B_8^{23}+B_9^{12})+u_9(B_7^{23}+B_{14}^{31})$$
$$+\boxed{u_{10}}(B_7^{33}+B_8^{22}+B_9^{22}+B_{14}^{33}+B_{15}^{11}+B_{16}^{11})$$
$$+\boxed{u_{11}}(B_9^{23}+B_{16}^{31})+u_{14}(B_{14}^{23}+B_{15}^{12})+u_{15}(B_{15}^{23}+B_{16}^{12})=0$$

$$\boxed{u_6}(B_9^{31}+B_{10}^{12})+\boxed{u_7}(B_{10}^{23}+B_{11}^{12})+\boxed{u_{10}}(B_7^{23}+B_{16}^{31})$$
$$+\boxed{u_{11}}(B_9^{33}+B_{10}^{22}+B_{11}^{22}+B_{16}^{33}+B_{17}^{11}+B_{18}^{11})$$
$$+u_{14}(B_{11}^{23}+B_{18}^{31})+u_{15}(B_{16}^{23}+B_{16}^{12})+u_{16}(B_{17}^{23}+B_{18}^{12})=0$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} \\ \alpha_{31} & 0 & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

連立方程式の導出



u_1
u_2
u_3
u_4
u_5
u_6
u_7
u_8
u_9
u_{10}
u_{11}
u_{12}
u_{13}
u_{14}
u_{15}
u_{16}
①
②
③
④
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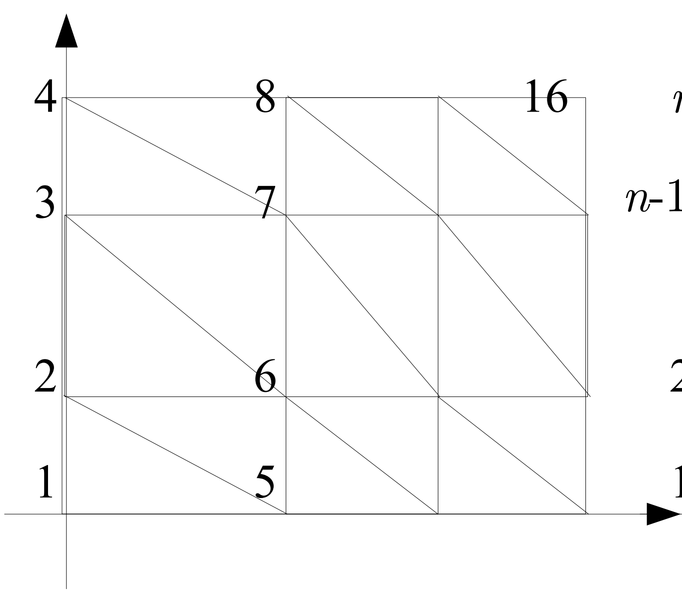
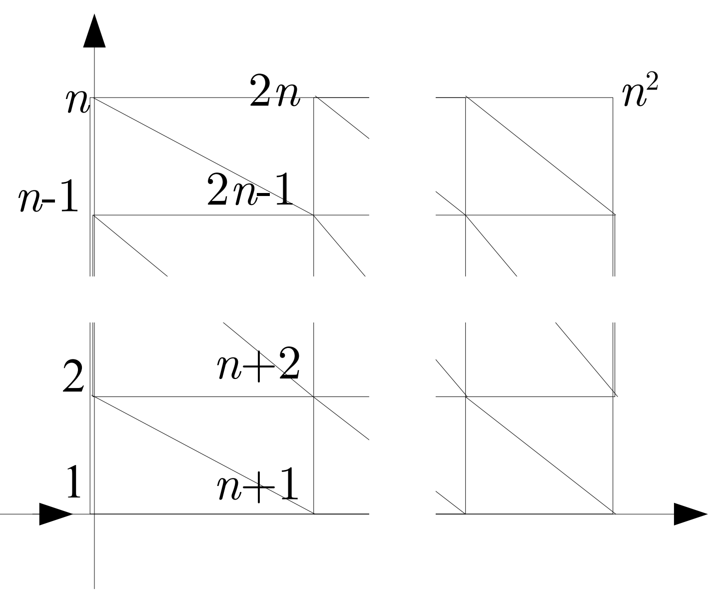
⑪
⑫
⑬
⑭
⑮
⑯
⑰
⑱

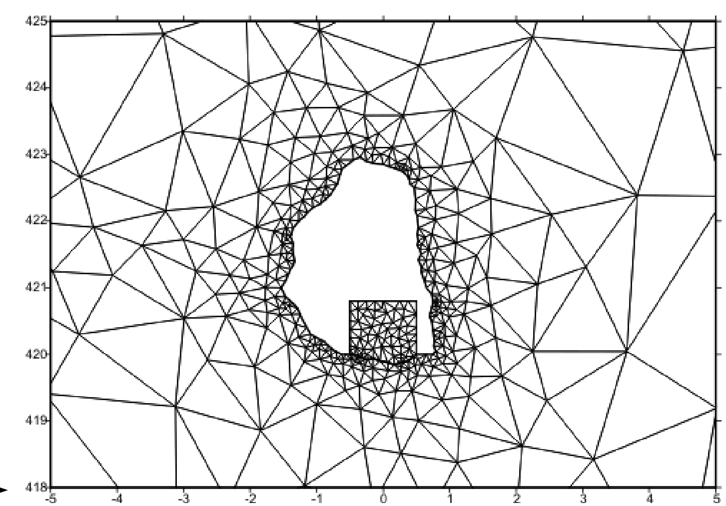
レポート(7)

学籍番号･氏名を記し提出してください。

- 次の三角形分割でLaplace方程式の境界値問題を考えた場合、有限要素法の係数行列はどのようになりますか？







授業レポート用紙：氏名（　　　　　　　　　　）学籍番号(　　　　　　　　)

2019年12月2日（月）

できれば授業の感想も書いてください。