授業スケジュール

第01回(09/30):熱伝導方程式の導出
第02回(10/07):差分法の基礎
10/14 体育の日
第03回(10/21):差分法の解法
10/28 休講
11/11 学生祭
第04回(11/12):熱伝導方程式の解法
第05回(11/18):有限要素法の準備
第06回(11/25):有限要素法の基底関数
第07回(12/02):有限要素法の連立方程式
第08回(12/09):有限要素法と変分法
第09回(12/16):スペクトル法と代用電荷法1
第10回(12/23):スペクトル法と代用電荷法2
01/13 成人の日
第11回(01/20):差分法演習
第12回(01/27):有限要素法演習
第13回(02/03):代用電荷法演習
第14回(02/10):課題演習
第15回(02/13):まとめ.... 木曜日ですが月曜授業です。

工学部本館8階 810室
学科教育用計算機システムで演習

ユーザID/パスワードが必要です。

計算科学特論

スペクトル法と代用電荷法2

代用電荷法によるLaplace方程式の解法

- 試行関数には電荷を模した基本解を用いる:
  厳密に調和関数になる

2次元 $$u \sim U(z) = \sum_{\mathrm{j}} c_{\mathrm{j}} \log |z - x_{\mathrm{j}}|$$

3次元 $$u \sim U(z) = \sum_{\mathrm{j}} c_{\mathrm{j}} / |z - x_{\mathrm{j}}|$$

- 試験関数（重み）にはデルタ関数を用いる
  （選点法を用いることと同等）

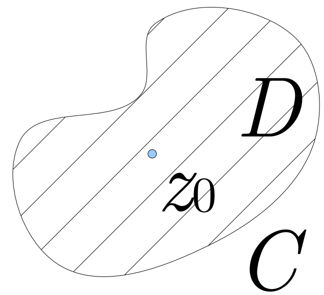
$$\int \ldots \int \delta(z - y_{\mathrm{k}})(u(z) - \sum_{\mathrm{j}} c_{\mathrm{j}} u_{\mathrm{j}}(z)) = u(y_{\mathrm{k}}) - U(y_{\mathrm{k}}) = 0$$

- 残差の計算に積分等は不要になる

代用電荷法の特徴

- Laplace方程式に適合したスペクトル法の一種で、Laplace方程式を厳密に満たす
- 境界条件だけで重みを定めるため数値積分が不要
- 展開に用いる関数をうまく選べば高精度
- 展開に用いる関数の準備＝電荷点の撰択が難しい
- 境界条件が滑らかでない問題の近似精度が下がる
- 非線形の問題を扱うことが難しい

代用電荷法

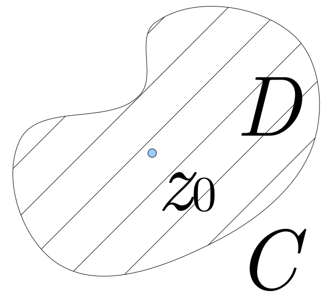


調和関数$g(z)$の境界値問題

$$z \in C \Rightarrow g(z) = b(z), \quad z \in D \Rightarrow \triangle g(z) = 0$$

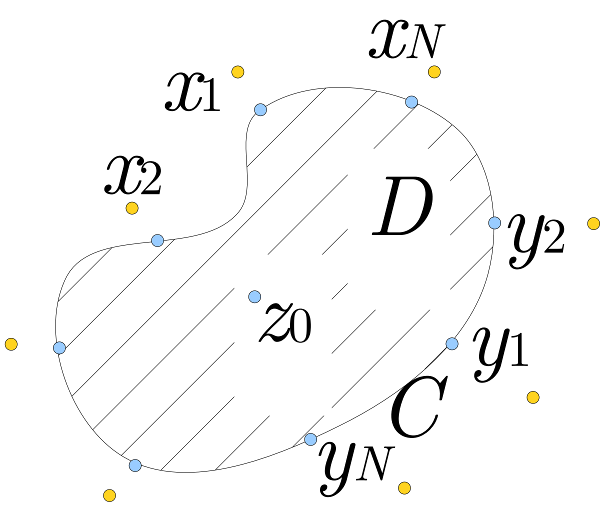
代用電荷法

Laplace方程式
の境界値問題



$z \in D \Rightarrow \triangle g(z) = 0$

$z \in C \Rightarrow g(z) = b(z)$



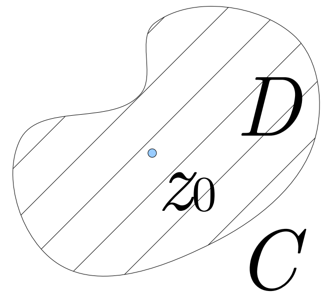
$x_1, \ldots, x_N \notin C \cup D$ ：電荷点

$$g(z) \approx G(z) = G_0 + \sum_{j=1}^{N} Q_j \log |z - x_j|$$

$\Rightarrow \triangle G(z) = 0$を満たす

代用電荷法

Laplace方程式
の境界値問題



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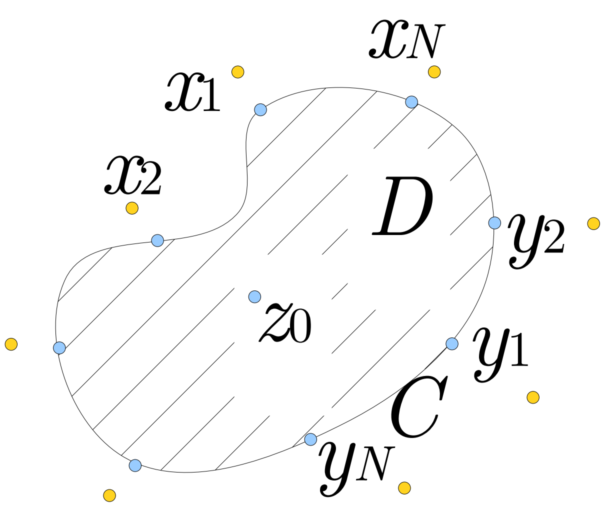

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$\Rightarrow \triangle G(z) = 0$を満たす

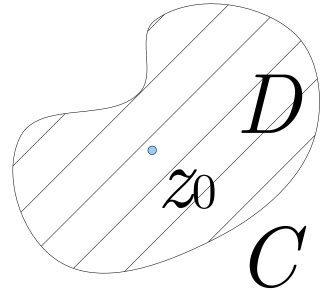
$y_1, \ldots, y_N \in C$ ：拘束点

選点的境界条件
＝拘束条件

$$g(y_k) \approx G(y_k) = b(y_k)$$
$$= G_0 + \sum_{j=1}^{N} Q_j \log|y_k - x_j|$$

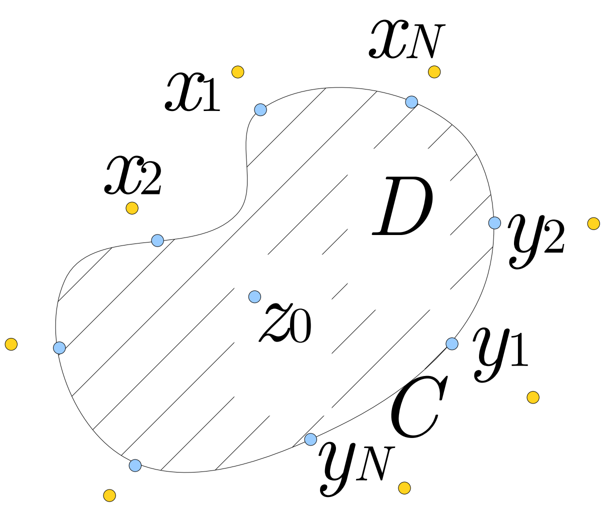
代用電荷法

Laplace方程式
の境界値問題



$z \in D \Rightarrow \triangle g(z) = 0$

$z \in C \Rightarrow g(z) = b(z)$



$x_1, \ldots, x_N \notin C \cup D$:電荷点

$g(z) \approx G(z) = G_0 + \sum_{j=1}^{N} Q_j \log|z - x_j|$

$\Rightarrow \triangle G(z) = 0$を満たす

$y_1, \ldots, y_N \in C$:拘束点

選点的境界条件
$=$拘束条件

$g(y_k) \approx G(y_k) = b(y_k)$

$= G_0 + \sum_{j=1}^{N} Q_j \log|y_k - x_j|$

不変性条件 $\sum_{j=1}^{N} Q_j = 0,$

[6]室田一雄, 代用電荷法におけるスキームの「不変性」について,
情報処理学会論文誌 **34**, 3, 533-535 (1993).

代用電荷法

拘束条件 $G_0+\sum_{j=1}^{N} Q_j \log|y_k-x_j|=b(y_k)$

不変性条件 $\sum_{j=1}^{N} Q_j = 0,$

$$\Updownarrow$$

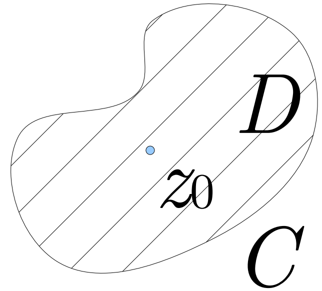
$$\begin{bmatrix} 1 & \log|y_1-x_1| & \cdots & \log|y_1-x_N| \\ \vdots & \ddots & \cdots & \vdots \\ 1 & \log|y_N-x_1| & \cdots & \log|y_N-x_N| \\ 0 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} G_0 \\ Q_1 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} b(y_1) \\ \vdots \\ b(y_N) \\ 0 \end{bmatrix}$$

連立一次方程式を解き $G_0,\ Q_1,\ldots,Q_N$を決めればよい

$$g(z)\approx G(z)=G_0+\sum_{j=1}^{N} Q_j \log|z-x_j|$$

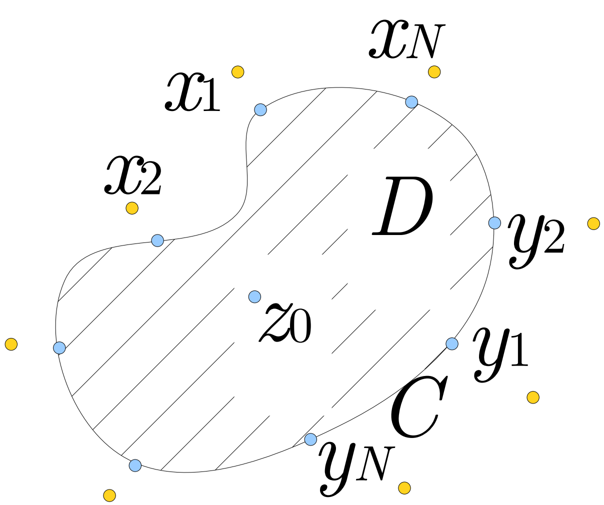
代用電荷法

Laplace方程式
の境界値問題



$z \in D \Rightarrow \triangle g(z) = 0$

$z \in C \Rightarrow g(z) = b(z)$



$$g(z) \approx G(z) = G_0 + \sum_{j=1}^{N} Q_j \log|z - x_j|$$

$x_1, \ldots, x_N \notin C \cup D$　：電荷点

$y_1, \ldots, y_N \in C$　：拘束点

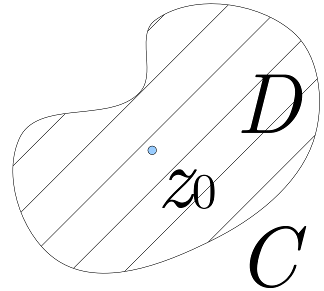
を置いて、

拘束条件 $G(y_k) = b(y_k)$ と不変性条件 $\sum_{j=1}^{N} Q_j = 0$,
を満たすように $G_0,\ Q_1, \ldots, Q_N$ を決めればよい。

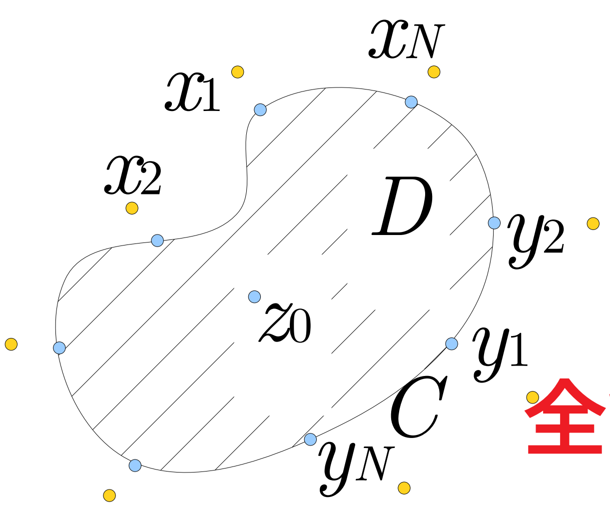
全ては電荷点･拘束点次第

代用電荷法

Laplace方程式
の境界値問題



$$z \in D \Rightarrow \triangle g(z) = 0$$
$$z \in C \Rightarrow g(z) = b(z)$$



$x_1, \ldots, x_N \notin C \cup D$ ：電荷点

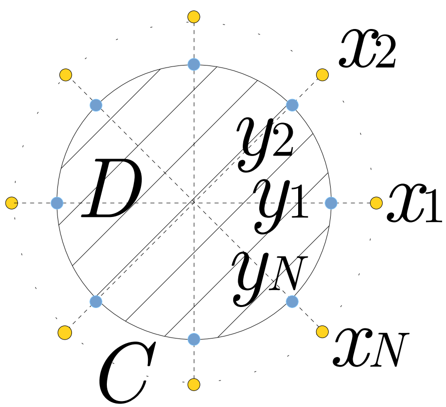
$y_1, \ldots, y_N \in C$ ：拘束点

全ては電荷点･拘束点次第

問題（$D,\ b(z)$）に適した電荷点･拘束点をどう選ぶ？

適切な電荷点･拘束点配置

円板領域に等角同相配置したとき



$C=\{z||z|=r\},\ D=\{z||z|<r\}$

$x_j=r\ \exp[\mathrm{i}2p(j\text{-}1)/N]\ j=1,\ldots,N,\ r>r,$

$y_k=r\ \exp[\mathrm{i}2p(k\text{-}1)/N]\quad k=1,\ldots,N.$

問題$\triangle g(z)=0$ in D, $g(z)=b(z)$ on C の厳密解 g が
Cを超えて$|z|<r_0$まで調和拡張できるとき、

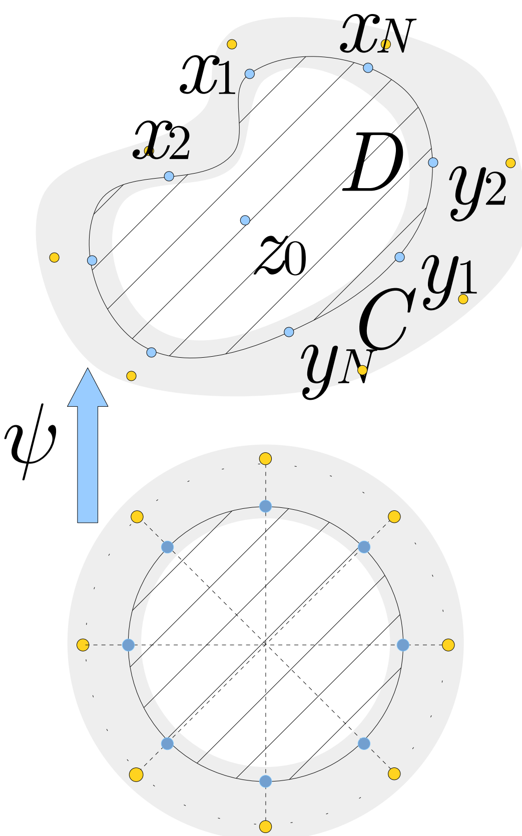
$\exists c>0,\ \tau>1$ s.t. $||g-G||_\infty \leq c\tau^{-N}$ 誤差の指数関数的減少

Gは代用電荷法による近似解, $\tau \simeq \min\{\rho/r, \sqrt{r_0/r}\}$

[7]Katsurada and Okamoto, A mathematical study of charge simulation method I, J. Fac. Sci. u-tokyo, SectIA, Math. **35**, 507-518 (1988)
[8]Katsurada, M., A mathematical study of charge simulation method II, J. Fac. Sci. u-tokyo, SectIA, Math. **36**, 135-162 (1988)

適切な電荷点･拘束点配置
~周辺等角写像ψによる方法~

周辺領域に等角写像した等角同相配置の点



単位円をCに写し､単位円を含む円環をCを含む帯状の周辺領域に写す等角写像ψで等角同相配置の点を写す;

$$x_j\!=\!\psi(\rho\omega^{j-1}),\ y_k\!=\!\psi(\omega^{k-1}),\ \rho\!>\!1,\ \omega\!=\!\mathrm{e}^{2\pi\mathrm{i}/N}.$$

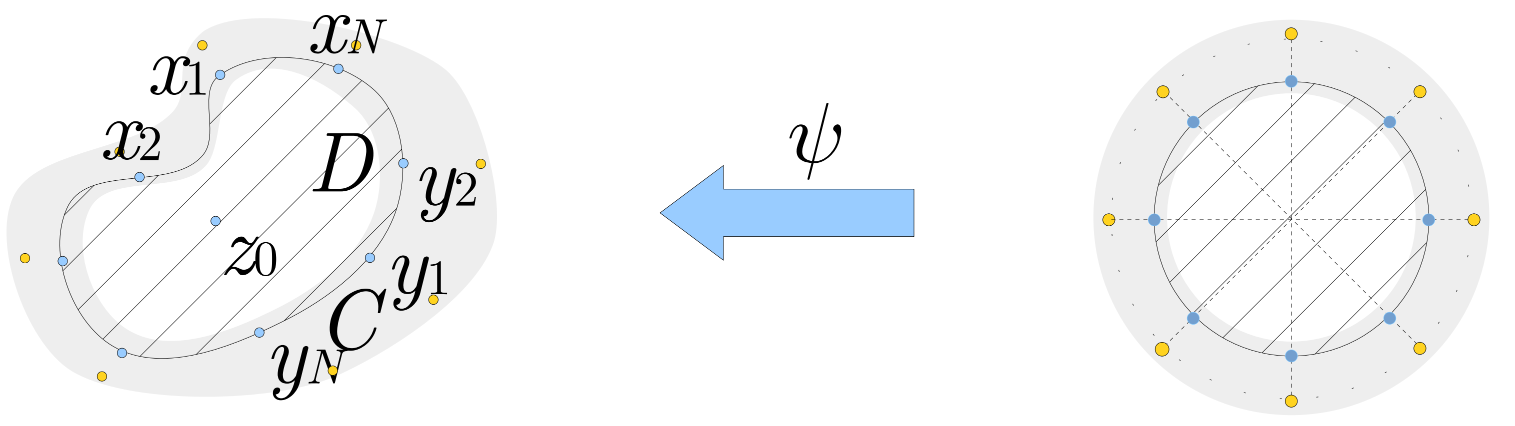
円板領域の等角同相配置と同様に代用電荷法による近似について､誤差の指数関数的減少が言える

$$\exists c\!>\!0,\ \tau\!>\!1\ \text{s.t.}\ ||g\!-\!G||_\infty\!\leq\! c\tau^{-N}$$

[9]Katsurada and Okamoto, The collocation points of the fundamental solution method for the potential problem, Comp. Math. Applic. **31**, 1, 123-137 (1996).

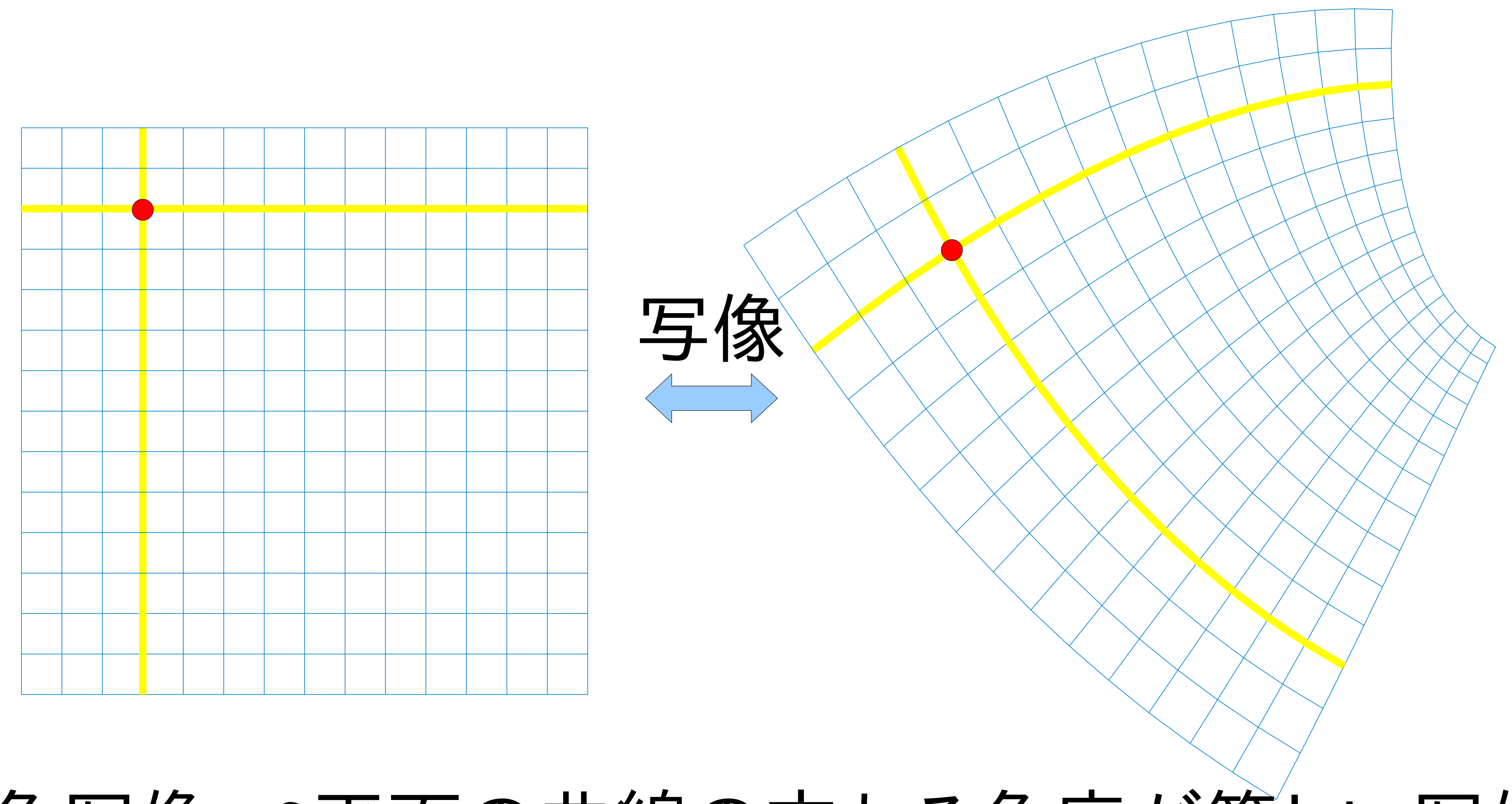
適切な電荷点·拘束点配置
~周辺等角写像ψによる方法~

周辺領域に等角写像した等角同相配置の点



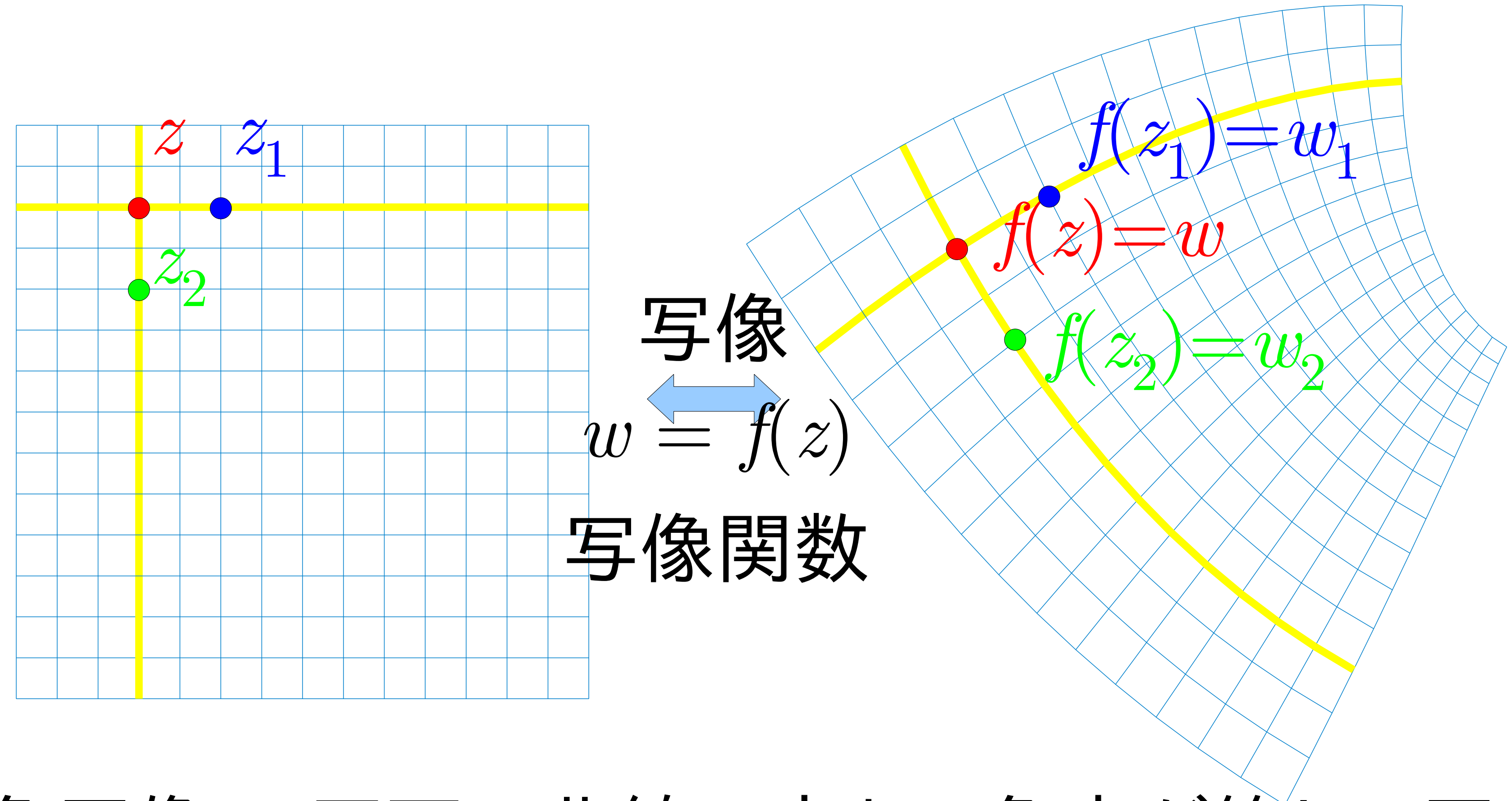
ψを求める→領域の境界値問題を解く必要がある

2次元平面の等角写像



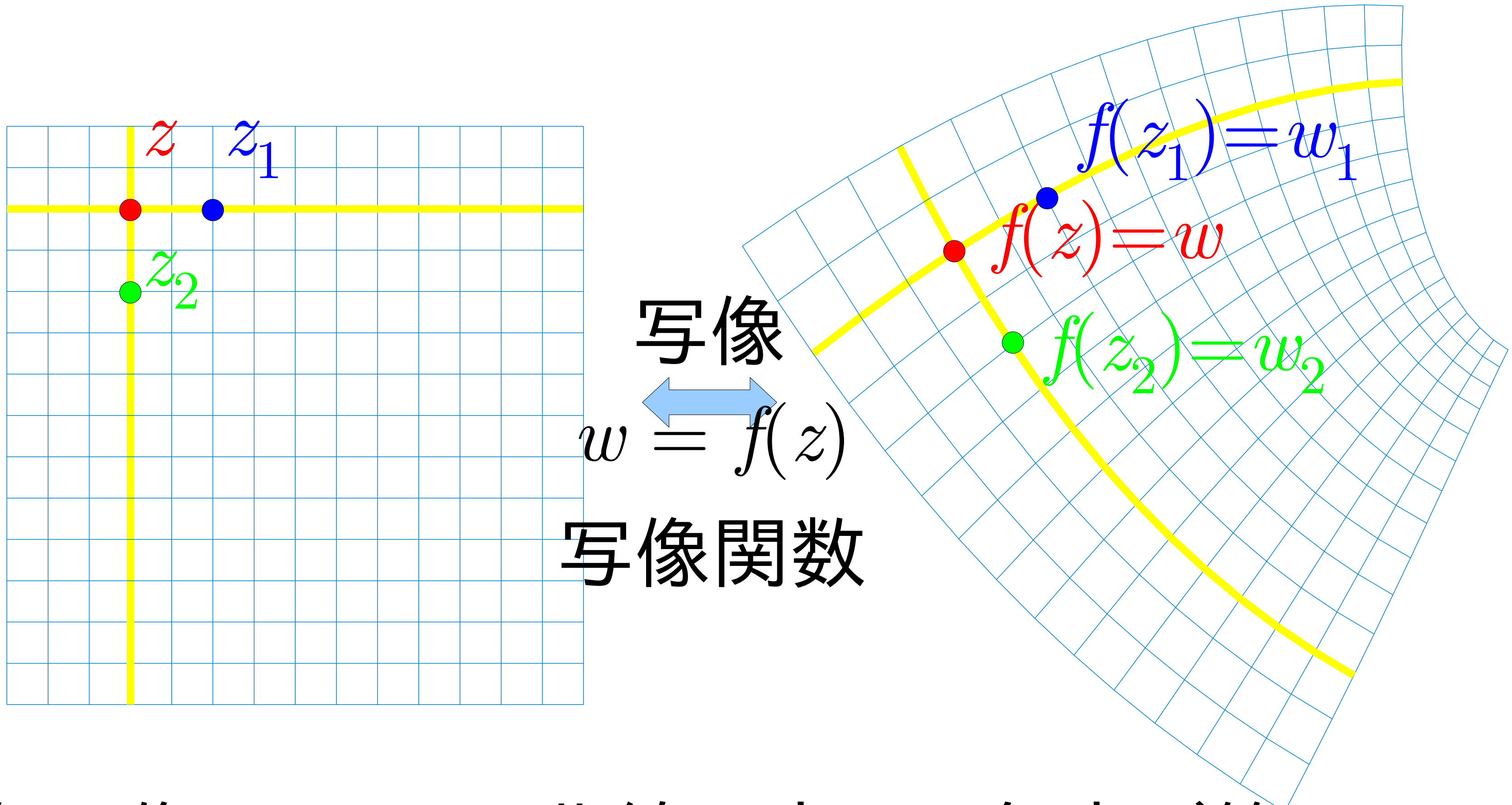
等角写像: 2平面の曲線の交わる角度が等しい写像

複素関数の等角写像



等角写像: 2平面の曲線の交わる角度が等しい写像

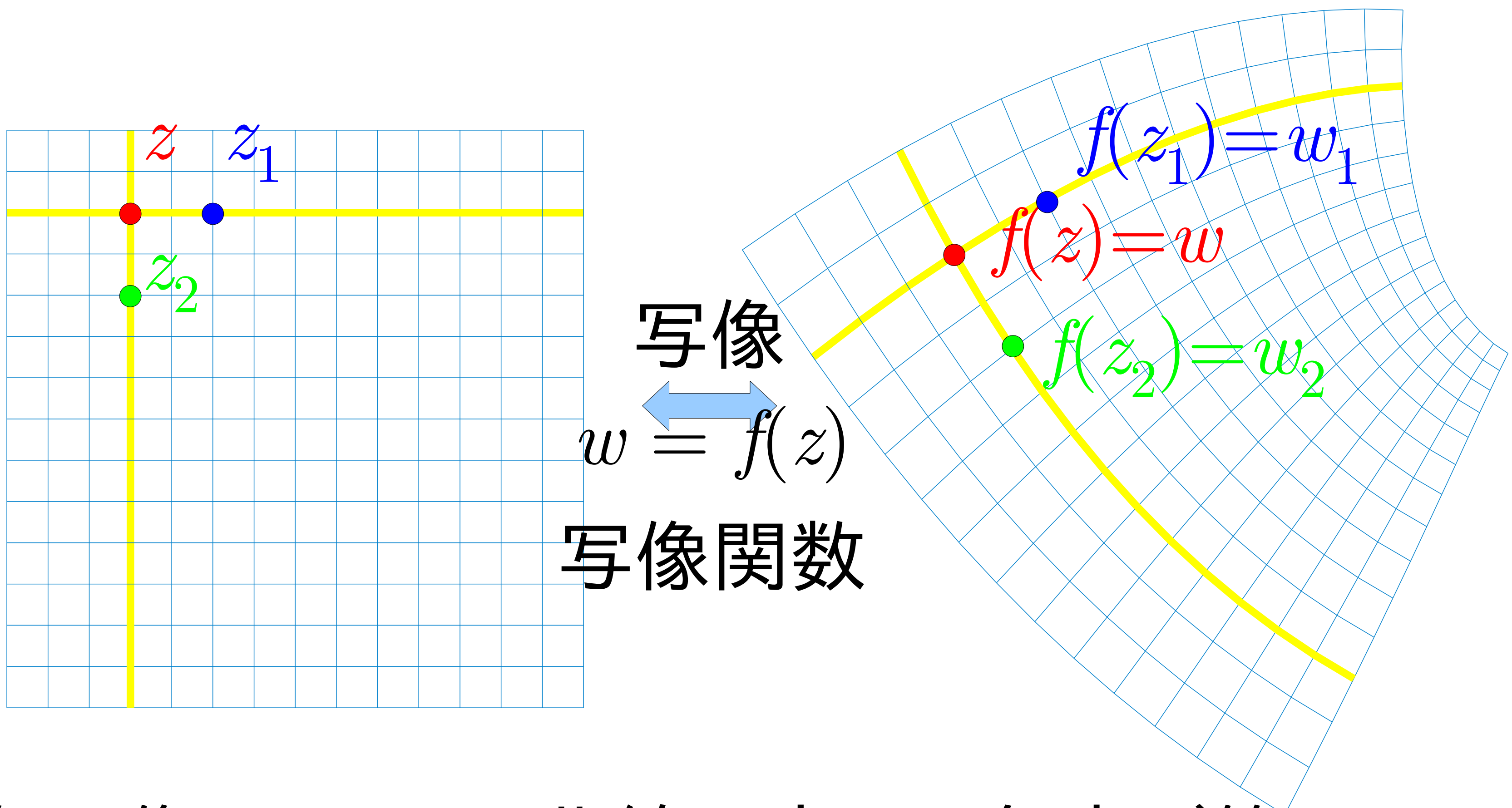
複素関数の等角写像



等角写像: 2平面の曲線の交わる角度が等しい写像

z に近い z_1, z_2 で $\dfrac{z_1-z}{z_2-z}=\dfrac{w_1-w}{w_2-w} \Rightarrow \lim_{z_1\to z}\dfrac{f(z_1)-f(z)}{z_1-z}=\lim_{z_2\to z}\dfrac{f(z_2)-f(z)}{z_2-z}$

複素関数の等角写像



等角写像: 2平面の曲線の交わる角度が等しい写像

$f(z)$が解析関数（微分できる関数）⇔ $w=f(z)$は等角写像

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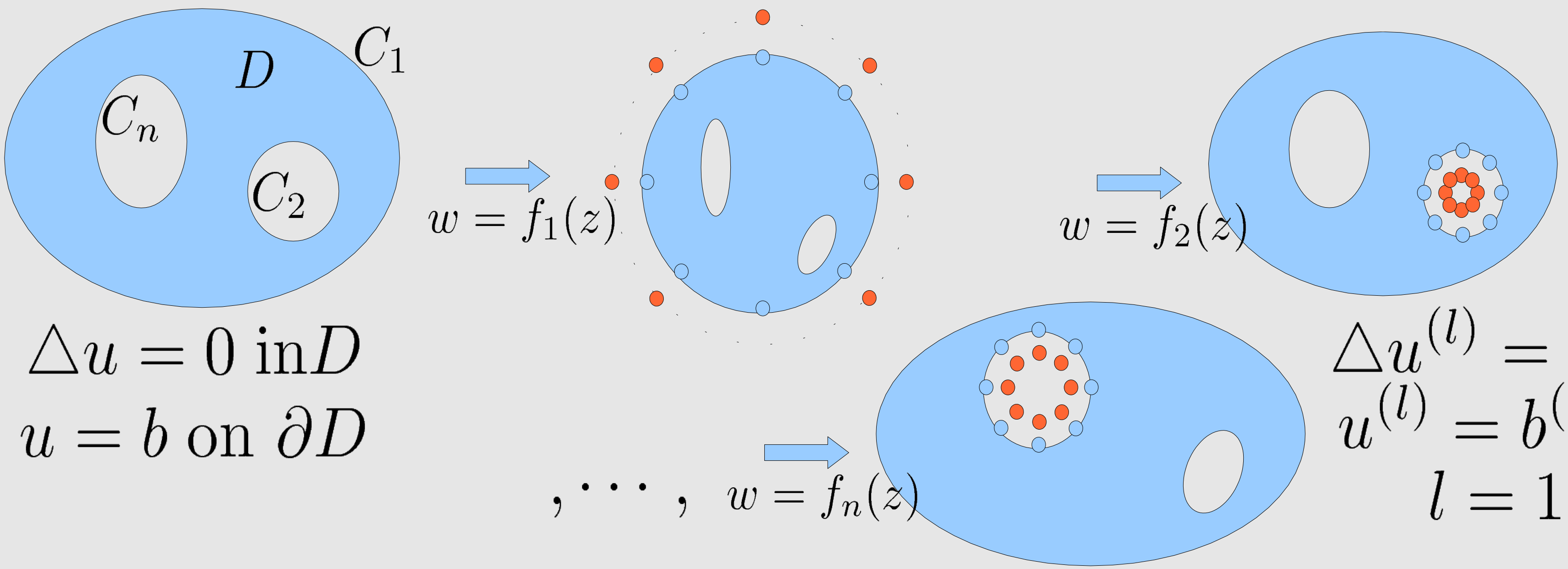
Charge and Collocation Points for Multiply-Connected Domain

D : multiply connected domain bounded by $C_1 \ldots, C_n$



$$\triangle u = 0 \text{ in} D$$
$$u = b \text{ on } \partial D$$

$$\triangle u^{(l)} = 0 \text{ in} D^{(l)}$$
$$u^{(l)} = b^{(l)} \text{ on } \partial D^{(l)}$$
$$l = 1, \ldots, n$$

$$u(z) \sim U_N(z) = \sum_l \sum_j Q_j^l \log |f_l(z) - x_j^{(l)}|.$$

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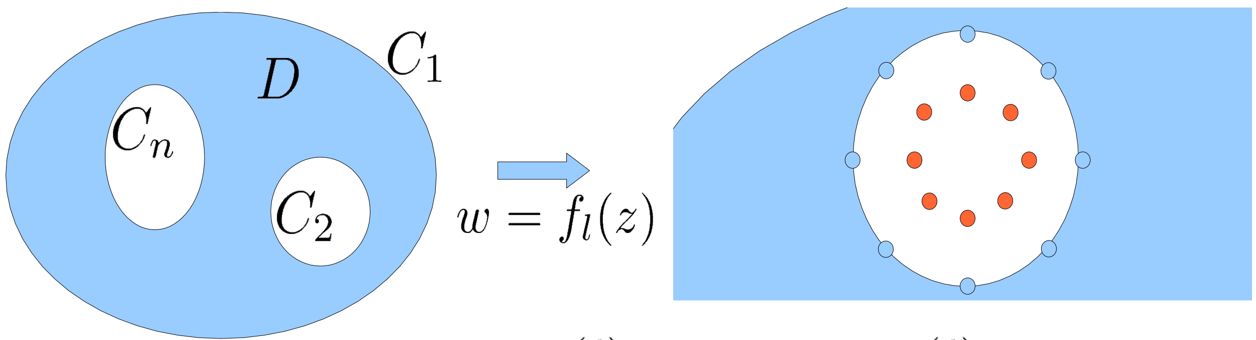
Exponential Error Decay



charge/coll. pts. are equally placed on the concentric circles

$$\triangle u = 0 \text{ in } D \qquad \triangle u^{(l)} = 0 \text{ in } D^{(l)}$$
$$u = b \text{ on } \partial D \qquad u^{(l)} = b^{(l)} \text{ on } C^{(l)} \qquad l = 1, \ldots, n$$

$$u(z) \sim U_N(z) = \sum_l \sum_j Q_j^l \log |f_l(z) - x_j^{(l)}|.$$

if harmonic func. $u^{(l)}$ is extensible beyond $C^{(l)}$

$$\Rightarrow \exists A^{(l)} > 0, \tau^{(l)} > 1, \quad \max_{C_l} |u - U_N| \leq A^{(l)} \tau^{(l)-N}.$$

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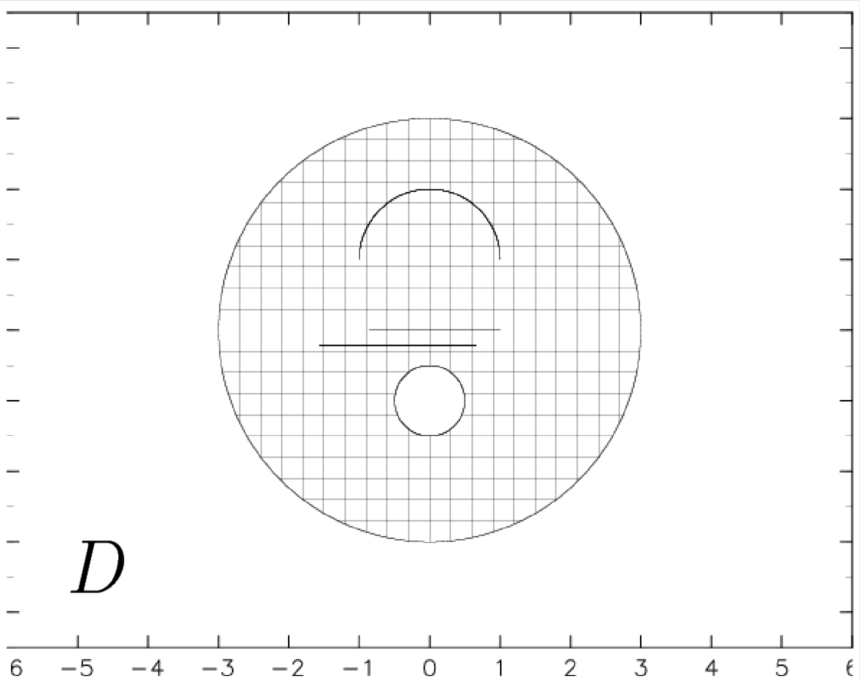
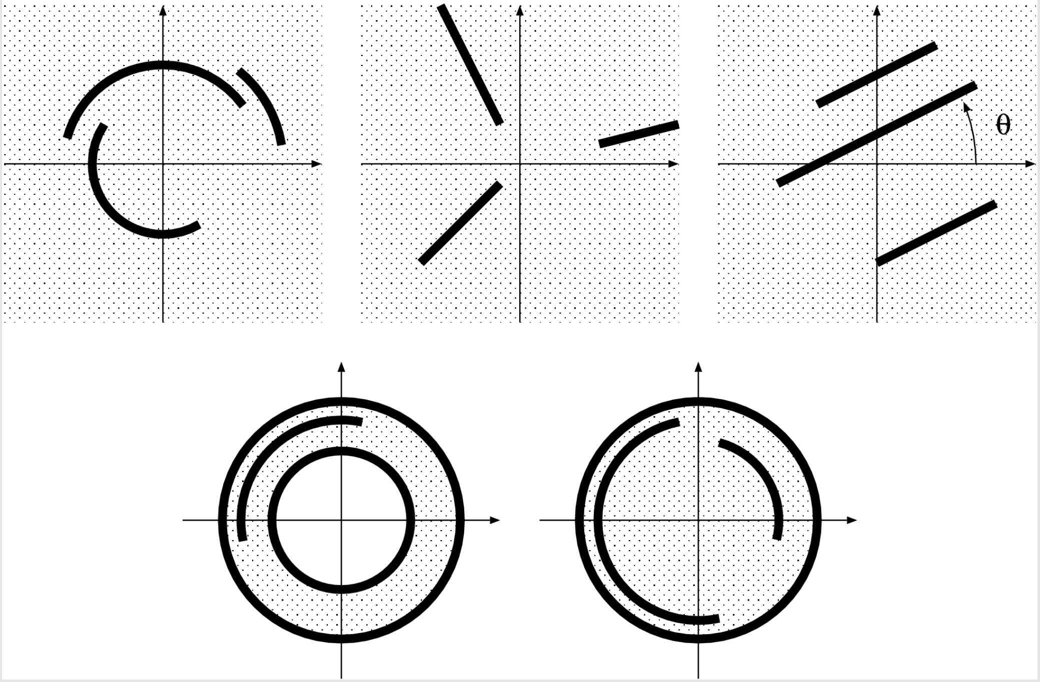
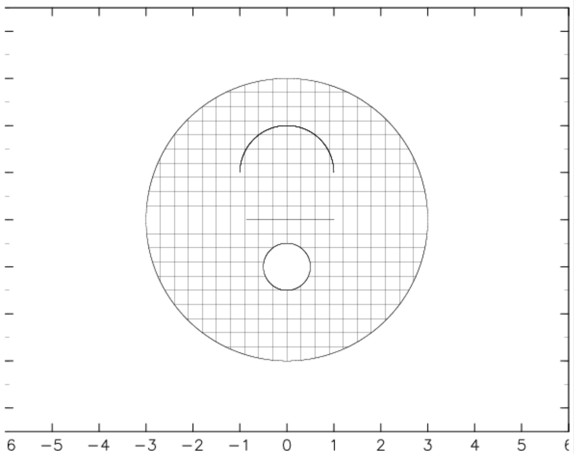
Numerical Example

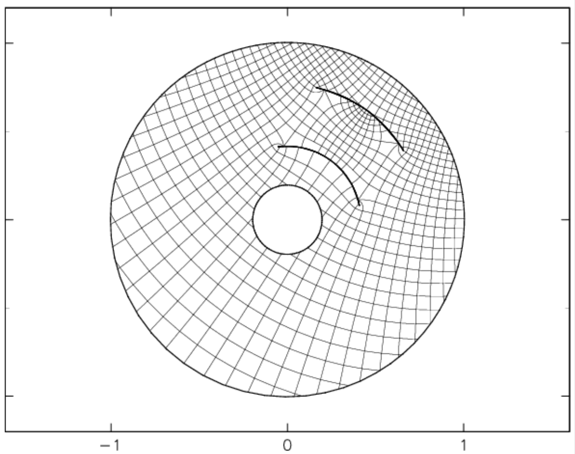


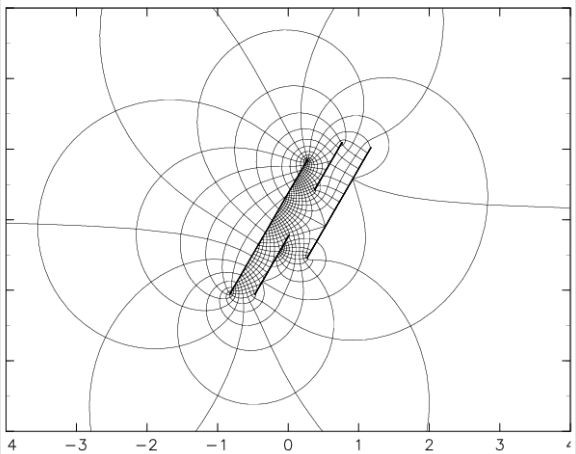


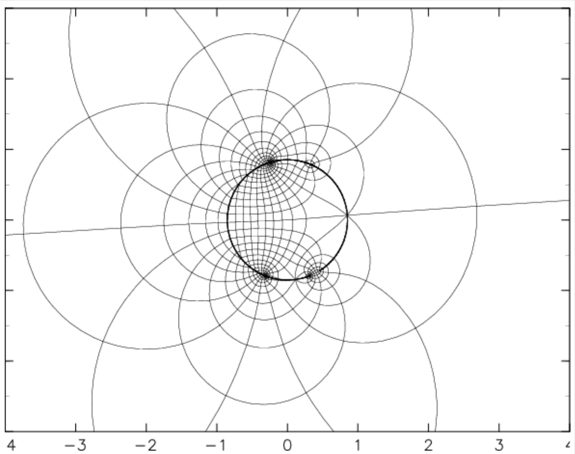
Conformal maps from problem domain with two slits onto Nehari's 5types of canonical slit domains

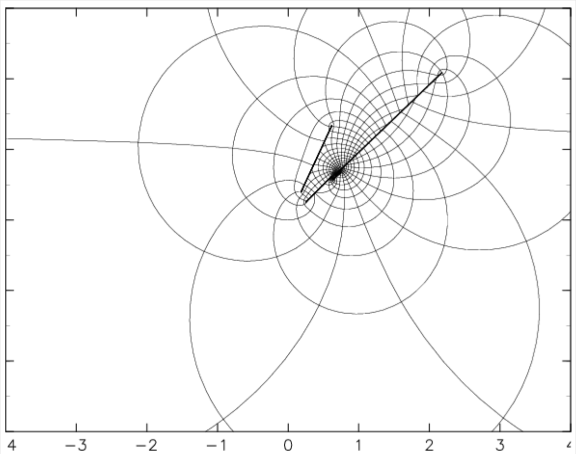
Numerical Example











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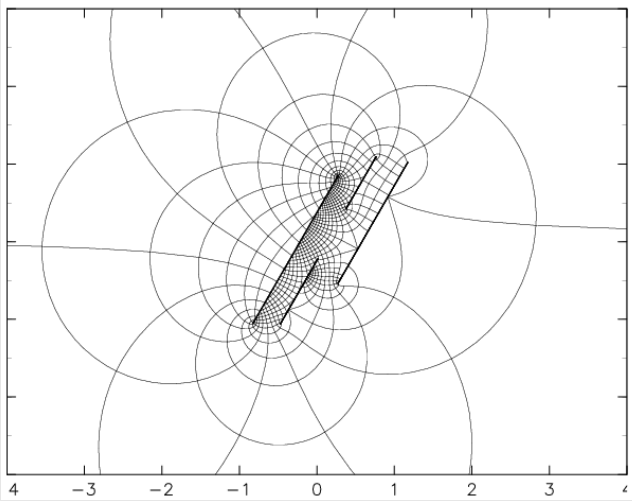
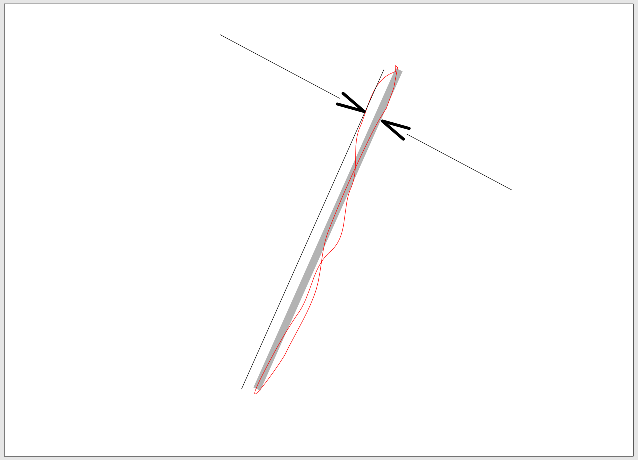
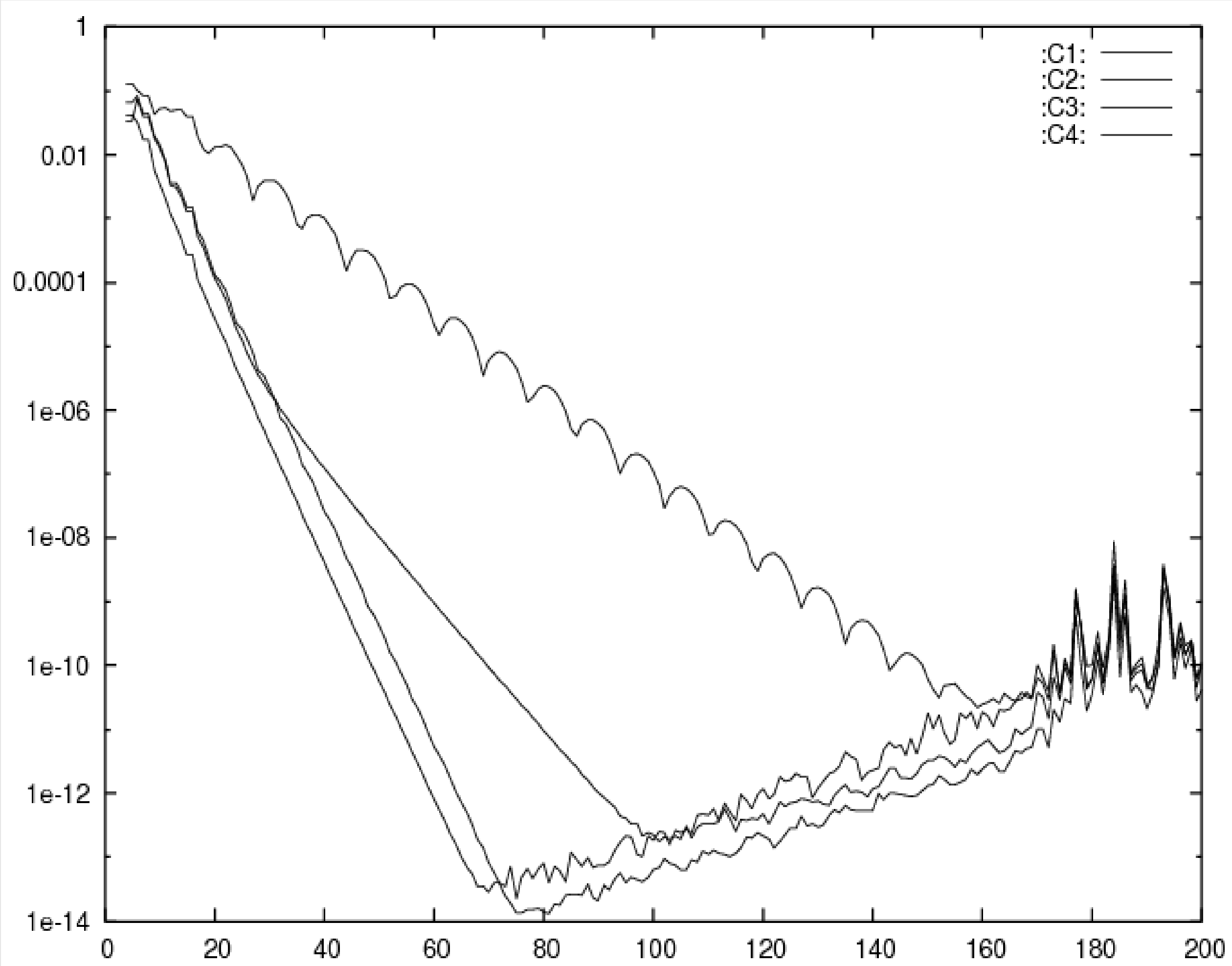
Numerical Example







errors estimated by deviation from slits

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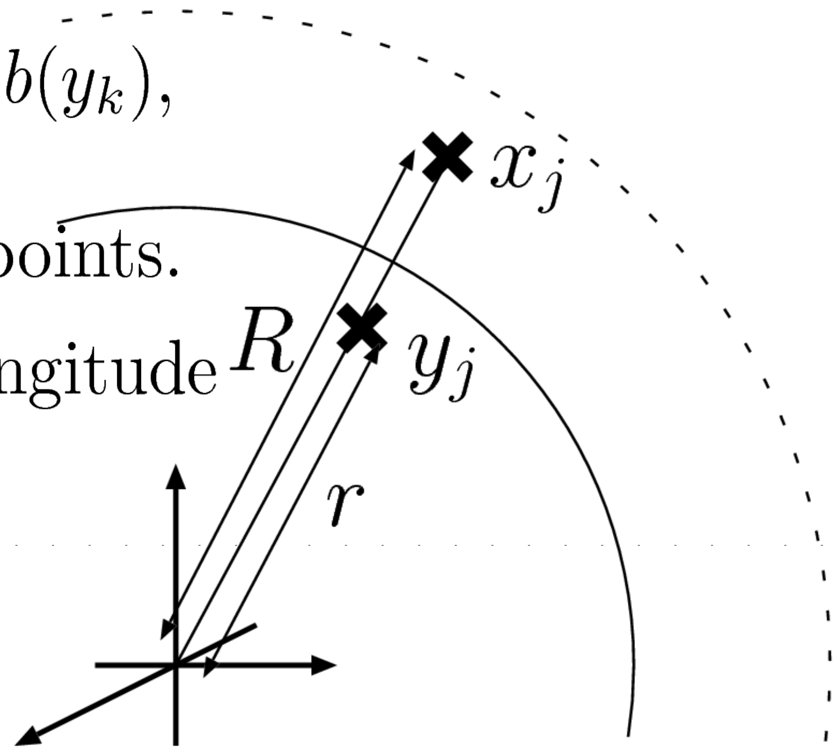
Applying to the 3D potential problems (sphere)
Ogata, Okano, Sugihara, and Amano (2003)

$$\triangle u = 0 \quad \text{in } D \subset \mathbb{R}^3, \quad u = b \quad \text{on } \partial D, \quad D = \{z \mid |z| < r\}.$$

$$U_N(z) = \sum_j Q_j / |z - x_j|, \, U_N(y_k) = b(y_k),$$

x_j : charge points, y_k : collocation points.

x_j and y_j has same latitude and longitude on concentric spheres for $j = 1, \ldots$



$\Rightarrow$ coeff. matrix is non-singular, approximation is available.

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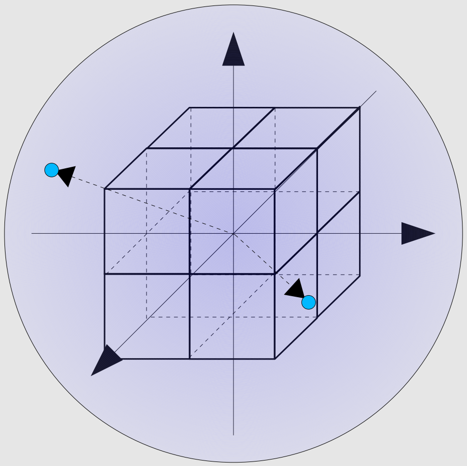
Uniform Points Arrangement and Exponential Error Decay

We examine the efficiency of ″equally″ placed charge and collocation points via some numerical experiments.

Candidates:

1. Generalized spiral points ($C = 3.6$)
2. Minimal energy points of Sloan and Womersley
3. Vertices of regular polyhedra
4. Points lifted onto sphere from the surface lattice points of cube



Generalized spiral points

″generalized spiral points″

$$(\theta_1, \phi_1), \ldots, (\theta_N, \phi_N)$$

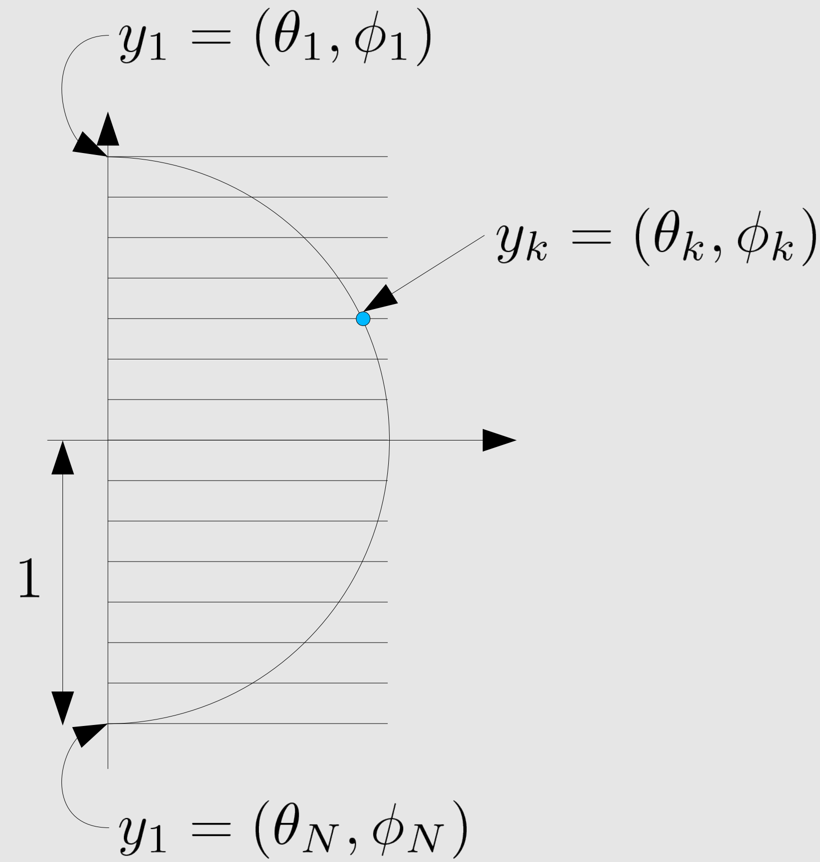
$$h_k = 1 - 2\frac{k-1}{N-1} \quad k = 1, \ldots, N$$

$$\theta_k = \arccos(h_k)$$

$$\phi_1 = \phi_N = 0$$

$$\phi_k = \phi_{k-1} + \frac{C}{\sqrt{N_0}} \frac{1}{\sqrt{1-h_k^2}}$$

$$k = 2, \ldots, N-1.$$



E. A. Rakhmanov, E. B. Saff and Y. M. Zhou: “Minimal discrete energy on the sphere,” Mathematical Research Letters **1** (1994), 647--662

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Generalized spiral points

"generalized spiral points"

$$(\theta_1, \phi_1), \ldots, (\theta_N, \phi_N)$$

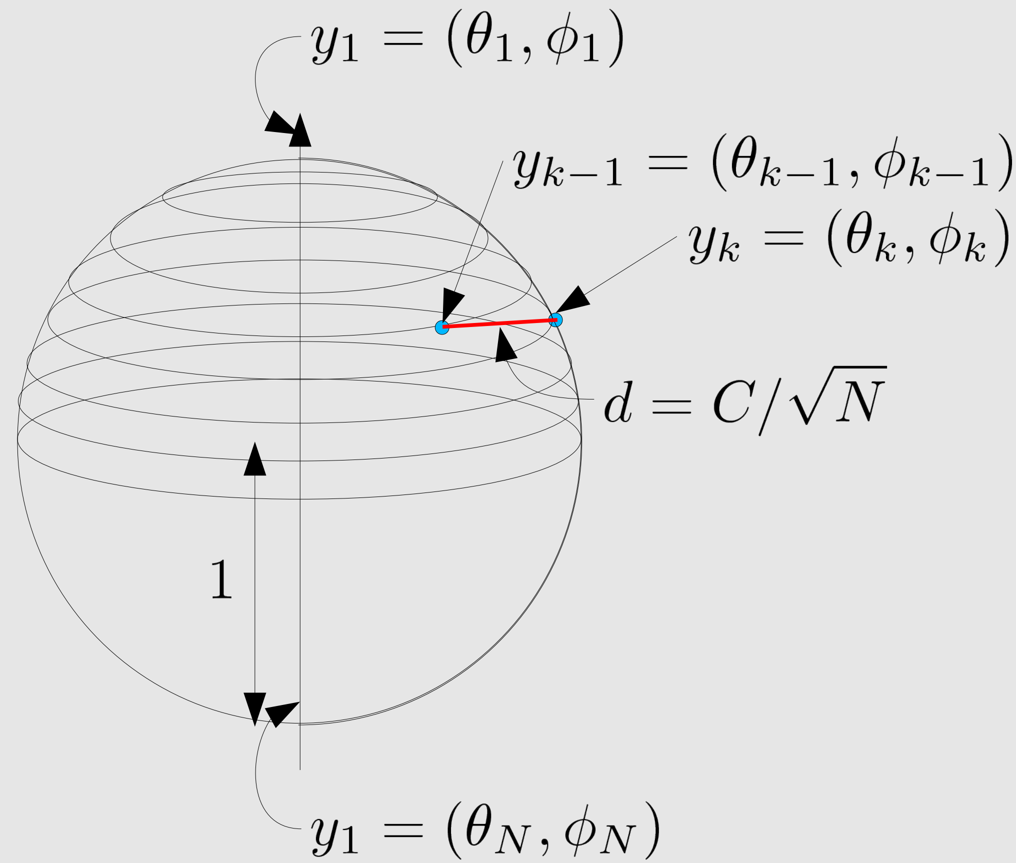
$$h_k = 1 - 2\frac{k-1}{N-1} \quad k = 1, \ldots, N$$

$$\theta_k = \arccos(h_k)$$

$$\phi_1 = \phi_N = 0$$

$$\phi_k = \phi_{k-1} + \frac{C}{\sqrt{N_0}} \frac{1}{\sqrt{1-h_k^2}}$$

$$k = 2, \ldots, N-1.$$



E. A. Rakhmanov, E. B. Saff and Y. M. Zhou: "Minimal discrete energy on the sphere," Mathematical Research Letters **1** (1994), 647--662

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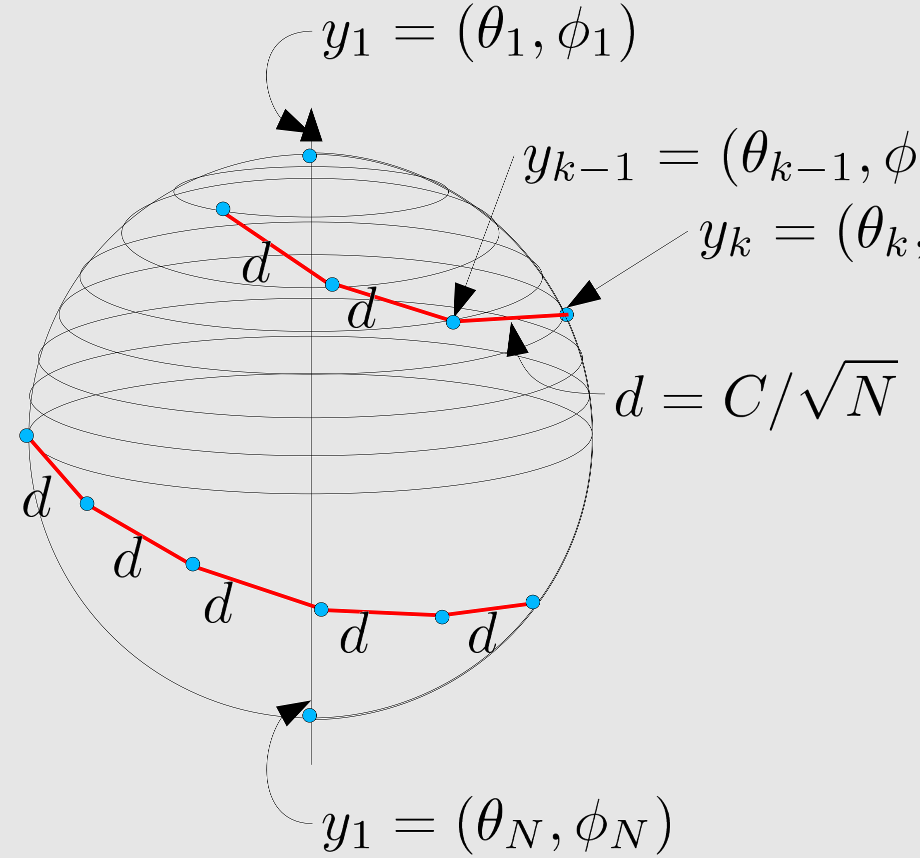

$$d = \frac{C}{\sqrt{N}} < \frac{4}{\sqrt{N}}$$

g.s.pts.: $y_1, \dots, y_N$ with $C = 3.6$ $\quad \epsilon(y_1, \dots, y_N) \sim E(N)$

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Generalized spiral points

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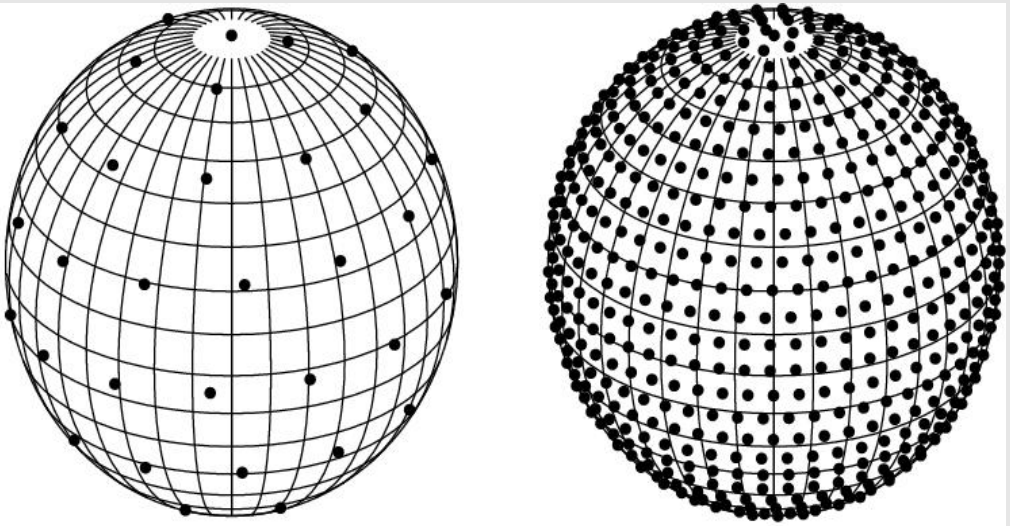

Fig.1: Generalized Spiral Points on Sphere
N=64(left), N=1000(right)

$$d = \frac{C}{\sqrt{N}} < \frac{4}{\sqrt{N}}$$

g.s.pts.: $y_1, \ldots, y_N$ with

$$C = 3.6 \qquad \epsilon(y_1, \ldots, y_N) \sim E(N)$$

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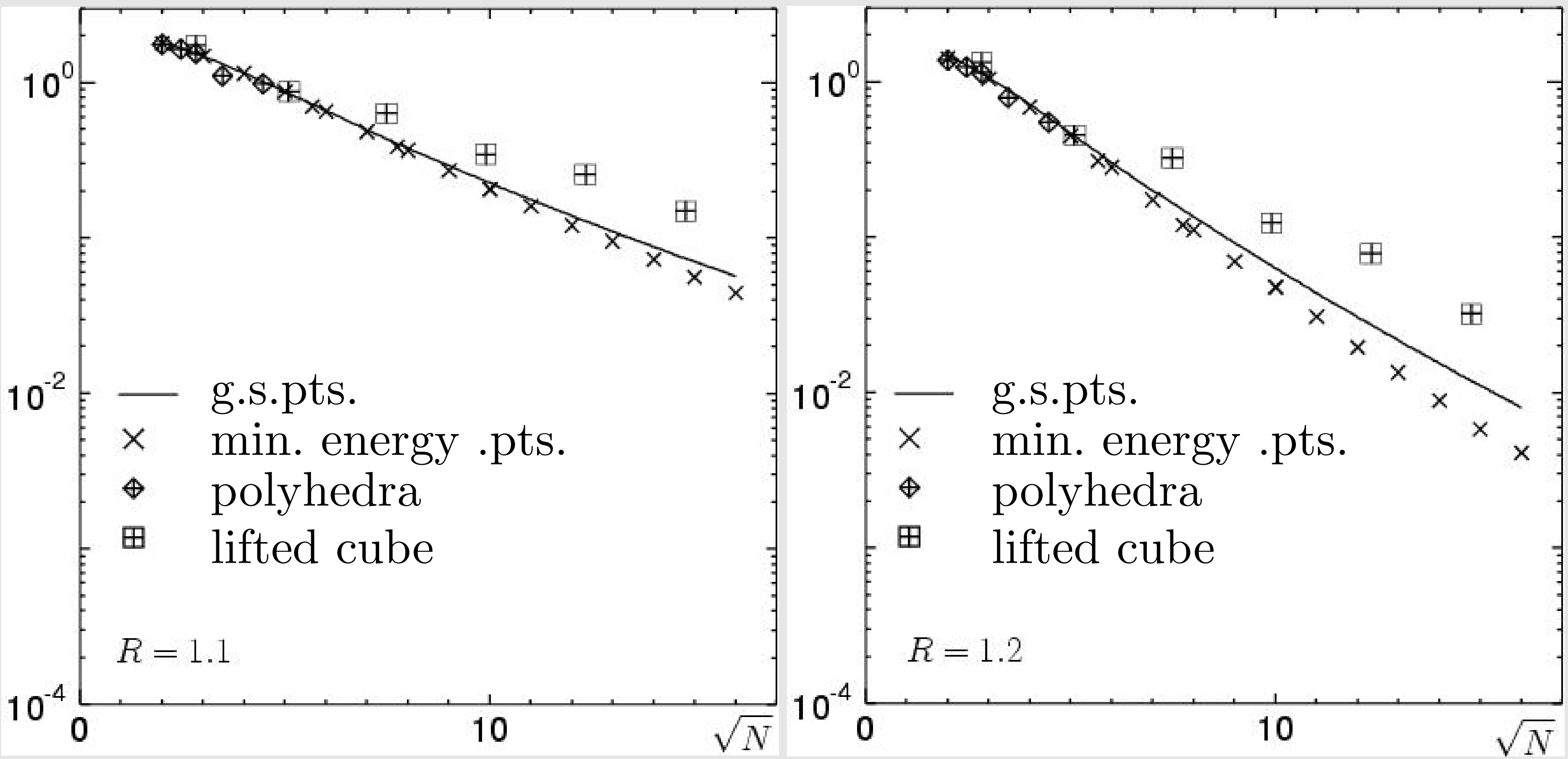
Numerical Examples

exponential decay of error



$$u(r, \theta) = r \cos \theta$$

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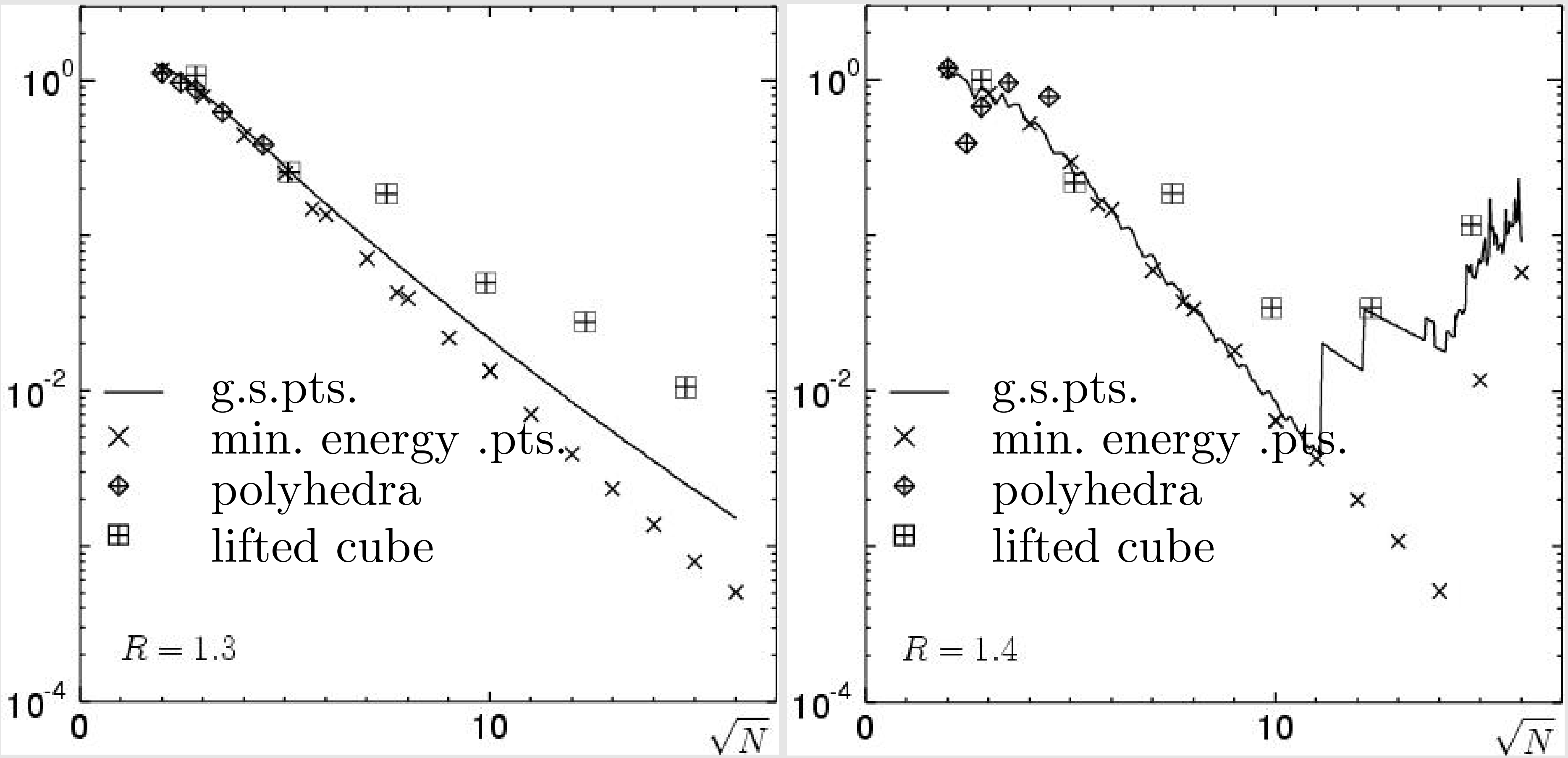
Numerical Examples

exponential decay of error



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Numerical Examples

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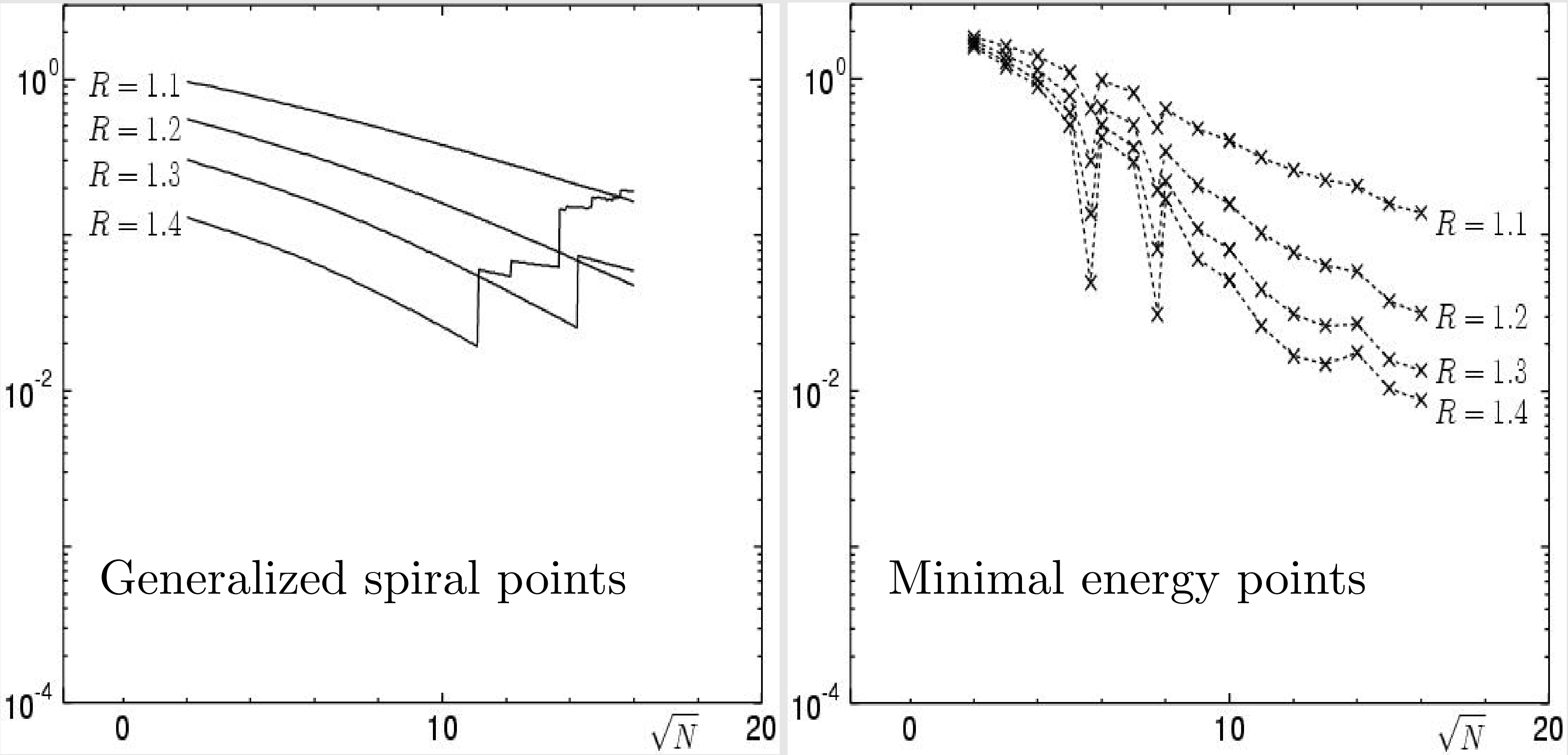

$$u(z) = |z - z_0|^{-1}, z_0 = (r = 1.5, \theta = 0)$$

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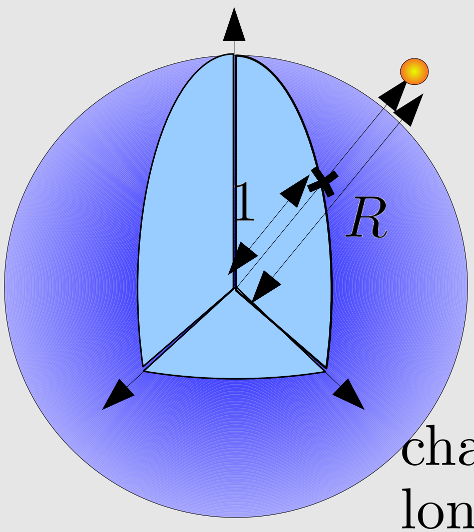
Exponential error decay on sphere

exponential decay of error for potential problems on sphere

$$\sup_{|z|<1} |u(z) - U_N(z)| \le c\tau^{-\sqrt{N}}, 0 < c, 1 < \tau = \tau(R, u, \dots)$$



by equally placed collocation points,
minimal energy points,
generalized spiral points,

and

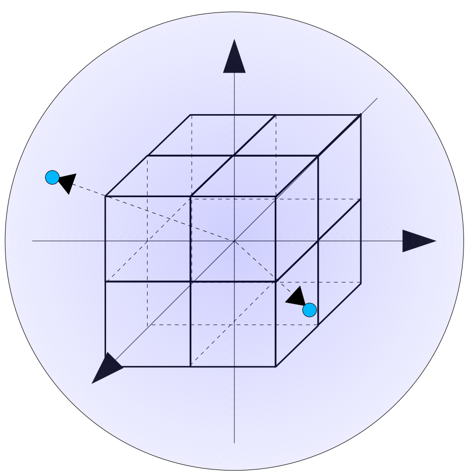
charge and collocation points of same latitude and longtitude on concentric spheres

$$x_j = Ry_j, \quad j = 1, \dots, N, R > 1$$

レポート(10)

学籍番号・氏名を記し提出してください。

- 立方体側面の等分点を元に球面上の「一様」な拘束点を用意すると、その点数はどのような増え方をしますか？



ヒント:

頂点8個＝6面×4×(1/3)

面の中央点＝6面×1と
辺の等分点＝6面×4×(1/2)
を加えると　26個

その次は？

※ 増やし方は色々考えられます。

授業レポート用紙：氏名（　　　　　　　　　　）学籍番号(　　　　　　　　　　)

2019年12月23日（月）

できれば授業の感想も書いてください。